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Dynamical System Synchronization

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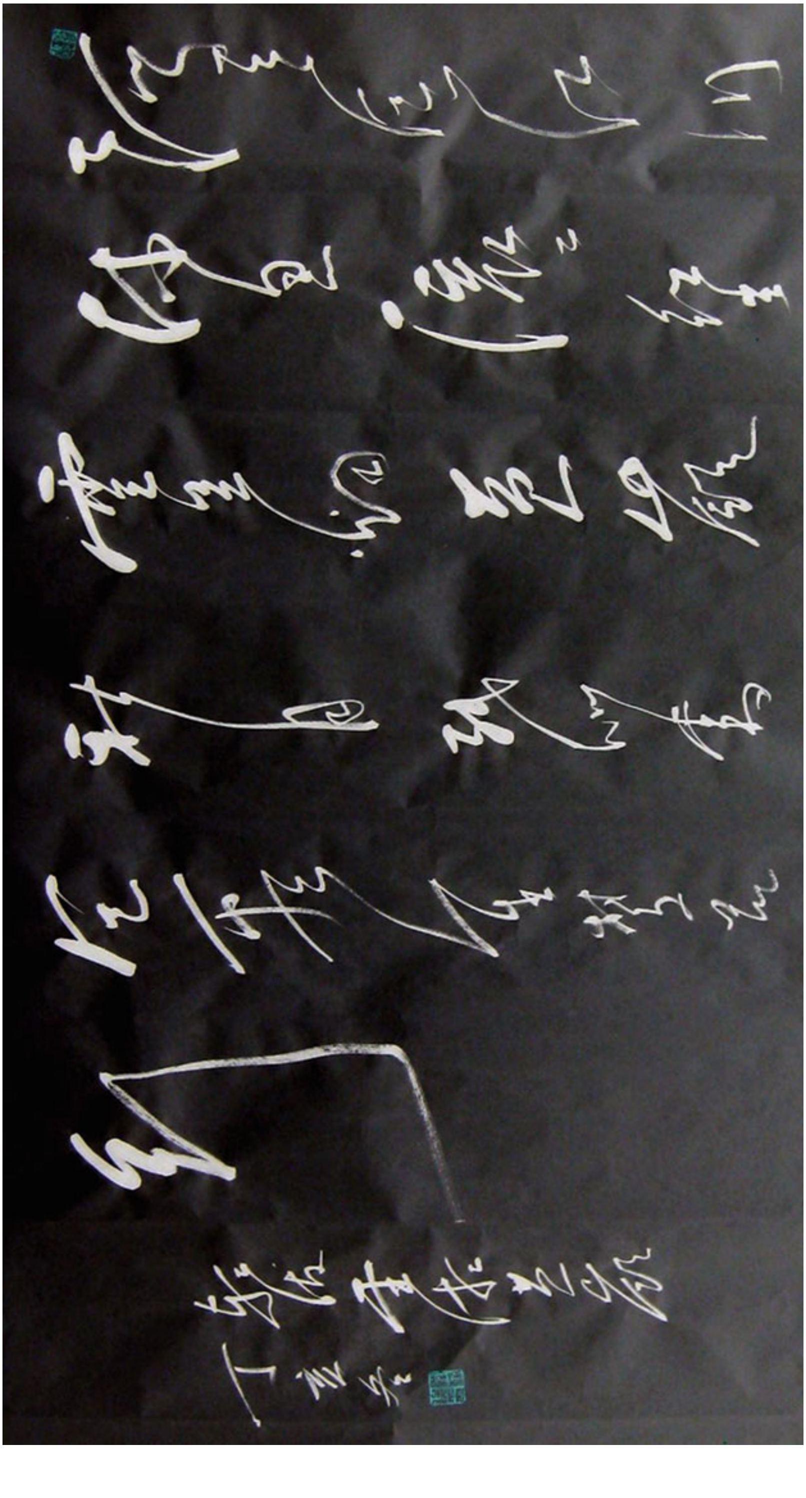

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Dynamical System Synchronization

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Preface

This book presented a theory of dynamical systems synchronization from a different point of view. Such synchronization theory is based on the theory of discontinuous dynamical systems. The synchronization of discrete dynamical systems is based on the Ying-Yang theory of discrete dynamical systems. The objective of this book is to throw out the different points of view to look into dynamical systems synchronization without the Lyapunov stability method.

This book consists of six chapters. In Chap. 1, a brief history of dynamical systems synchronizations is presented. The current methods for dynamical systems synchronization are mainly based on the Lyapunov stability theory. Thus, the dynamical systems for synchronization should be similar systems, which is far away for practical applications. To solve such difficulty, a theory of dynamical systems synchronization with specific constraints is presented in this book from the theory of discontinuous dynamical systems. In Chap. 2, the switchability of a flow to the boundary in discontinuous dynamical systems is presented in order to help one understand the synchronization theory of two dynamical systems with specific constraints. In Chap. 3, the basic concepts of dynamical systems synchronization are presented first, and the theory of dynamical systems synchronization with a specific constraint is presented. For a further development of dynamical systems synchronization, the theory of two dynamical systems with multiple constraints is discussed in Chap. 4. In Chap. 5, the function synchronization of two distinct dynamical systems is discussed to show how to apply the theory of dynamical systems synchronization to practical problems. In Chap. 6, the theory for discrete dynamical systems synchronization is presented from the Ying-Yang theory of discrete dynamical systems.

Finally, I would like to appreciate my students (Yu Guo and Fuhong Min) for completing numerical computations. Herein, I thank my wife (Sherry X. Huang) and my children (Yanyi Luo, Robin Ruo-Bing Luo, and Robert Zong-Yuan Luo) for tolerance, patience, understanding, and support. This is what I can bring them for happiness.

Edwardsville, IL, USA Albert C. J. Luo

Contents

Chapter 1
Introduction

With human-being development and progress, coordinate systems are used in order that the characteristics and behaviors of everything in nature can be described quantitatively. In other words, because the coordinates systems are used extensively, one gradually understands and improves the objective world. Similarly, to describe the complexity of a changing process of a thing with time, one often adopts a given or known process to compare with such a changing process with time. Further, one obtains the similarity, instantaneous similarity and differences between the two dynamical processes for a time interval, and one determines the complexity of a dynamical system to another known dynamical system. Such similarity in a certain time interval is a kind of synchronization. In addition, the synchronization of two or more dynamical systems is a basis to understand an unknown dynamical system from one or more well-known dynamical systems. In other words, the response complexity of an unknown system to one or more well-known systems can be measured and compared through such synchronicity. Thus, the synchronization in dynamical systems should be treated as an important concept, and the concept of "*synchronization*" is also a universal concept for dynamical systems. Based on the aforesaid reasons, in this book, a theory of synchronization of dynamical systems will be presented as a theoretic frame work.

A theory for synchronization of multiple dynamical systems under specific constraints was developed from a theory of discontinuous dynamical systems in Luo [1]. The concepts on synchronization of two or more dynamical systems to specific constraints were given. The synchronization, desynchronization and penetration of multiple dynamical systems to multiple specified constraints were discussed, and the necessary and sufficient conditions for such synchronicity were developed. The synchronicity of two dynamical systems to a single specific constraint and to multiple specific constraints was discussed, and the synchronization and the corresponding complexity for multiple slave systems with multiple master systems were presented. The meaning of synchronization for dynamical systems with constraints is extended as a generalized, universal concept. The theory presented in this book may be as a universal theory for dynamical systems. The book provides a theoretic frame work in order to control the slave systems which

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can be synchronized with master systems through specific constraints in a general sense. The theory of dynamical systems synchronization will be presented in this book, which is not relative to the Lyapunov stability theory, and the corresponding conditions for synchronicity are necessary and sufficient conditions for synchronicity of dynamical systems with specific constraints.

1.1 A Brief History

The investigation on synchronization should return back to the seventeenth century. Huygens [2] gave a detailed description of the synchronization of two pendulum clocks with a weak interaction. In fact, Huygens looked into two modal shapes of vibration. If the coupled pendulums possess small oscillations with the same initial conditions or the initial phase difference is zero, the two pendulums will be synchronized. If the initial phase difference is 180°, observed is the antisynchronization of two pendulums. For a general case, the motion of the two pendulums will be combined by the synchronization and anti-synchronization modes of vibration. So far, one focuses on four classes of synchronizations of two or more dynamical systems (1) identical or complete synchronization, (2) generalized synchronization, (3) phase synchronization, (4) anticipated and lag synchronizations and amplitude envelope synchronization. All the synchronizations of two or more systems at least possess one constraint for synchronicity, and such synchronizations experience the characteristics of asymptotic stability. Once the two or more systems form a state of synchronization for a specific constraint, such a state should be stable. The detailed discussion on such issue can be referred to Pikovsky et al. [3] and Boccaletti [4].

After the Huygens's investigation, Rayleigh worked on the theory of sound, which describes synchronization in acoustic systems in Rayleigh [5]. In 1920s, the synchronization was stimulated by the development of electrical and radio wave propagations. For an early investigation on synchronizations, one focused on the limit cycles in self-excited dynamical systems, resonance phenomena in multiple-degrees of freedom systems and, steady-state motion in forced vibration. The limit cycle in self-excited dynamical systems was discussed (e.g., [6]), which is a kind of synchronization and such synchronization can be stabilized. The other discussions on steady-state motion and resonance in nonlinear oscillations can be referred in many books (e.g., [7, 8]). Recently, one tried to control a flow of dynamical systems with attractors. Such an investigation is actually to look into a dynamical system synchronized with a goal dynamics, as discussed in Jackson [9].

For identical or complete synchronization of two systems, Pecora and Carroll [10] presented a criterion of the sub-Lyapunov exponents to determine the synchronization of two systems connected with common signals. The common signals are as constraints for such two systems. Based on this idea, the synchronized circuits for chaos were presented by Carroll and Pecaora [11]. Since then, one focused on developing the corresponding control methods and schemes to achieve the synchronization of two dynamical systems with constraints. Pyragas [12] presented two methods for chaos control with a small time continuous perturbation, which can achieve a synchronization of two chaotic dynamical systems. Kapitaniak [13] used such a continuous control to present the synchronization of two chaotic systems. Ding and Ott [14] pointed out a slave system (receiver system) is not necessary to be a replica of part of master systems. Rulkov et al. [15] discussed a generalized synchronization of chaos in directionally coupled chaotic systems. Kocarev and Parlitz [16] developed a general method to construct chaotic synchronized systems, which decomposes the given systems into the active and passive systems. Peng et al. [17] presented the chaotic synchronization of n-dimensional systems, and Pyragas [18] discussed the weak and strong synchronizations of chaos by the coupling strength of two dynamical systems. Ding et al. [19] provided a review on the control and synchronization of chaos in high-dimensional dynamical systems. In addition, Boccaletti et al. [20] presented an adaptive synchronization of chaos for secure communication. Abarbanel et al. [21] used a small force to control a dynamical system to given orbits. Pyragas [22] systematically introduced some basic ideas about the generalized synchronization of chaos. Yang and Chua [23] used linear transformations to investigate generalized synchronization. Zhan et al. [24] investigated the complete and generalized synchronizations of coupled time-delay systems. Campos and Urias [25] presented a mathematical description of multi-modal synchronization with chaos. The definition of master–slave synchronization was given and a multivalued, synchronized function was introduced. Koronovskii et al. [26] discussed the duration of a process of complete synchronization of two coupled, identical chaotic systems. The other investigations and applications on synchronization can be found in laser systems (e.g., [27–30]), human cardiorespiratory system (e.g., [31]). Mosekilde et al. [32] discussed chaotic synchronization and applied such concepts to living systems, and recent contributions on synchronization in biosystems can be found (e.g., [33–36]). Kocarev and Parlitz [37] investigated synchronizing spatiotemporal chaos in coupled nonlinear oscillators. Teufel et al. [38] presented the synchronization of two flow-excited pendula, which can recall Huygens' work [2]. Yamapi and Woafo [39] investigated synchronizations in a ring of four mutually coupled self-sustained electromechanical devices. Recently, other investigations on synchronization of coupled dynamical systems can be found (e.g., [40–43]). Newell et al. [44] investigated synchronization in chaotic diode resonator. Mbouna Ngueuteu et al. [40] investigated higher order nonlinearity on the dynamics and synchronization of two coupled electromechanical devices. Boccaletti et al. [45] gave a systematical review about the synchronization of chaotic systems. The definitions and concepts were further clarified. Chen et al. [46] gave a review on stability of synchronized dynamics and pattern formation in coupled systems. The dynamics and synchronization of coupled systems were investigated via control schemes (e.g., [47–50]). In addition to focusing on chaotic synchronization of continuous dynamical systems, one also has been interested in the synchronization of discrete systems with mappings. Pecora et al. [51] discussed the volume-preserving and volume-expanding synchronized chaotic systems through discrete maps. Stojanovski et al. [52] used the symbolic dynamics to investigate chaos synchronization, and information entropy was introduced to the synchronization of chaotic systems through discrete maps. Rulkov [53] discussed a regularization of synchronized chaotic bursts. Further, Afraimovich et al. [54] gave a mathematical investigation on the generalized synchronization of chaos in noninvertible maps. Barreto et al. [55] discussed the geometrical behavior of chaos synchronization through discrete maps. Hu et al. [56] investigated the hybrid projective synchronization of a general class of chaotic maps. Except for the complete and generalized synchronization of dynamical systems, another important synchronization is phase synchronization. As mentioned before, the phase synchronization exists in self-excited vibration systems, forced nonlinear vibrating systems and coupled nonlinear systems. For such investigation, the perturbation techniques were used (e.g., [7, 8]). Kuramoto [57] used the concept of phase synchronization (or entertainments) to investigate the waves and turbulence in chemical oscillations. Based on this concept, Zaks et al. [58] investigated imperfect phase synchronization through the alternative locking ratios. Feng and Shen [59] investigated phase synchronization and anti-phase synchronization of chaos in degenerate optical parametric oscillator. Pareek et al. [60] used multiple one-dimensional chaotic maps to investigate cryptography, and the extension of such a research can be found in Xiang et al. [61]. Bowong et al. [62] used the parameter modulation of a chaotic system for secure communications. Fallahi et al. [63] adopted the extended Kalman filter and multi-shift cipher algorithm for secure chaotic communication, and Kiani-B et al. [64] used fractional chaotic systems to secure communication through an extended fractional Kalman filter. Wang and Yu [65] used multiple-chaotic systems to develop a block encryption algorithm with a dynamical sequence. Soto-Crespo and Akhmediev [66] showed nonlinear synchronization and chaos through solitons as strange attractors. Hung et al. [67] discussed chaos synchronization of two stochastically coupled random Boolean networks. The more discussion about phase synchronization in oscillatory networks was presented in Osipov et al. [68]. The investigations on synchronization on the dynamical systems with time-delay were very active and the recent results can be found (e.g., [24, 33, 34, 62, 69–72]).

From the above-mentioned brief discussions of systems synchronization, it can be concluded that the synchronization of two or more dynamical systems is that the corresponding flows of the two or more dynamical systems are constrained under specific constraint conditions for a time interval. If the constraint conditions are considered as constraint boundaries, the synchronization of the two or more dynamical systems can be investigated by the theory of discontinuous dynamical systems. Luo [73] developed a theory for discontinuous dynamical systems, and the more detailed discussion was presented in Luo [1, 74–76]. In Luo [77], the theory for discontinuous dynamical systems was adopted to develop a theory for synchronization of dynamical systems with specific constraints. The concepts of dynamical systems synchronization with specific constraints were introduced. The necessary and sufficient conditions for the synchronization, desynchronization and penetration were developed. The synchronization complexity for multiple slave systems with multiple master systems was discussed under specific constraints. Using such a synchronization theory of two dynamical systems, in 2011, the synchronization dynamics of two distinct dynamical systems were presented without the Lyapunov method (e.g., [78–81]). The periodic and chaotic synchronizations of two distinct dynamical systems were presented. Recently, the function synchronization of two distinct dynamical systems was investigated from the theory of discontinuous dynamical systems (e.g., [82, 83]).

1.2 Book Layout

The main body in this book will discuss the synchronization of two dynamical systems from the theory of discontinuous dynamical systems, including four chapters, and the complete dynamics and synchronization of discrete dynamical systems will be presented in last chapter.

In Chap. 2, a general theory for the passability of a flow to a specific boundary in discontinuous dynamical systems will be presented. The concepts of real and imaginary flows will be presented. The G-functions for discontinuous dynamical systems will be presented to describe the general theory of the passability of a flow to the boundary. Based on the G-function, the passability of a flow from a domain to an adjacent one will be discussed. With the concepts of real and imaginary flows, the full and half, sink and source flows to the boundary will be discussed in detail. A flow to the boundary in a discontinuous dynamical system is either passable or non-passable. Thus, the switching bifurcations between the passable and non-passable flows will be presented.

In Chap. 3, the concepts on synchronization of two or multiple dynamical systems to specific constraints are presented, which is different from the idea of traditional synchronization of two dynamical systems. For such synchronization, Lyapunov stability method cannot be adopted. The synchronization, desynchronization and penetration of two or multiple dynamical systems to a specific constraint are discussed from the theory of discontinuous dynamical systems, and the necessary and sufficient conditions for such synchronicity will be investigated.

In Chap. 4, the synchronization for two dynamical systems to multiple constraints will be discussed, and the synchronization and the corresponding complexity for multiple slave systems with multiple master systems will be discussed briefly. As in Luo [77], the synchronicity of two dynamical systems with multiple constraints will be presented. The mathematical description of the synchronicity of two dynamical systems to multiple constraints will be given, and the corresponding necessary and sufficient conditions for the synchronicity of two dynamical systems to the constraints are presented.

In Chap. 5, the synchronization of two dynamical systems will be treated as a boundary in discontinuous dynamical systems, and such a boundary is time-varying. The boundary and domains for one of two dynamical systems are constrained by the other. The corresponding conditions for such synchronization will be presented via the theory for the switchability and attractivity of edge flows to the specific edges. The synchronization of two totally different dynamical systems will be presented as an application.

In Chap. 6, a set of concepts on “Ying” and “Yang” in discrete dynamical systems will be presented. Based on the Ying-Yang theory, the complete dynamics of discrete dynamical systems will be discussed for an understanding of dynamical behaviors. From the ideas of the Ying-Yang theory of discrete dynamical systems, the companion and synchronization of discrete dynamical systems will be presented herein, and the corresponding conditions are presented as an integrity part of dynamical system synchronization. The synchronization dynamics of Duffing and Henon maps will be discussed.

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Chapter 2
Discontinuity and Local Singularity

In this chapter, a general theory for the passability of a flow to a specific boundary in discontinuous dynamical systems will be discussed. The concepts of real and imaginary flows will be introduced. The *G*-functions for discontinuous dynamical systems will be presented to describe the passability of a flow to the boundary. Based on the *G*-function, the passability of a flow from a domain to an adjacent one will be discussed. With the concepts of real and imaginary flows, the full- and half-sink and source flows to the boundary will be discussed as well. A flow to the boundary in a discontinuous dynamical system can be passable or non-passable. Thus, all possible switching bifurcations between the passable and non-passable flows will be presented.

2.1 Discontinuous Dynamical Systems

For any discontinuous dynamical system, there are many vector fields defined on different domains in phase space, and such differences between two vector fields in two adjacent domains cause flows to be non-smooth or discontinuous at the boundary of the domains. To investigate the dynamics of discontinuous dynamical systems, consider a discontinuous dynamical system on a universal domain $\mho \subset \mathscr{R}^n$, and the passability of a flow from one domain to its adjacent domains will be discussed first. Thus, subdomains Ω_α $(\alpha \in I,\ I = \{1, 2, \cdots, N\})$ of the universal domain $\mho$ will be introduced and the vector fields on the subdomains may be defined differently. If there is a vector field on a subdomain, this subdomain is said to be an accessible domain. Otherwise, such a domain is said to be an inaccessible domain. Thus, the domain accessibility can provide a design possibility for discontinuous dynamical systems. The corresponding definitions of the domain accessibility are given as follows.

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Definition 2.1 A subdomain of a universal domain $\mho$ in a discontinuous dynamical system is termed an *accessible* subdomain if at least a specific, continuous vector field can be defined on such a subdomain.

Definition 2.2 A subdomain of a universal domain $\mho$ in a discontinuous dynamical system is termed an *inaccessible* subdomain if no any vector fields can be defined on such a subdomain.

Since the accessible and inaccessible subdomains exist in discontinuous dynamical systems, the universal domain $\mho$ is classified into the connectable and separable domains. The connectable domain is defined as follows.

Definition 2.3 A domain $\mho$ in phase space is termed *a connectable domain* if all the accessible subdomains of the universal domain can be connected without any inaccessible subdomain.

Similarly, a definition of the separable domain is given.

Definition 2.4 A domain is termed a *separable domain* if the accessible subdomains in the universal domain are separated by inaccessible domains.

Since any discontinuous dynamical system possesses different vector fields defined on each accessible subdomain, the corresponding dynamical behaviors in those accessible subdomains Ω_α are distinguishing. The different behaviors in distinct subdomains cause flow complexity in the domain $\mho$ of discontinuous dynamical systems. The boundary between two adjacent, accessible subdomains is a bridge of dynamical behaviors in two domains for flow continuity. Any connectable domain is bounded by the universal boundary $S \subseteq \mathscr{R}^r$ $(r = n - 1)$, and each subdomain is bounded by the subdomain boundary surface $S_{\alpha\beta} \subset \mathscr{R}^r$ $(\alpha, \beta \in I)$ with or without the partial universal boundary. For instance, consider an n-dimensional connectable domain in phase space, as shown in Fig. 2.1a through an n_1-dimensional, subvector $\mathbf{x}_{n_1}$ and an n_2-dimensional, subvector $\mathbf{x}_{n_2}$ $(n_1 + n_2 = n)$. The hatched area Ω_α is a specific subdomain, and the other subdomains are white. The dark, solid, closed curve represents the original boundary of the domain $\mho$. For the separable domain, there is at least an inaccessible subdomain to separate the accessible subdomains. The union of inaccessible subdomains is also called the "inaccessible sea." The inaccessible sea is the complement of the accessible subdomains to the universal (original) domain $\mho$. That is determined by $\Omega_0 = \mho \backslash \cup_{\alpha \in I} \Omega_\alpha$. The accessible subdomains in the domain $\mho$ are also called the "islands." For illustration of such a definition, a separable domain is shown in Fig. 2.1b. The thick curve is the boundary of the universal domain, and the gray area is the inaccessible sea. The white regions are the accessible domains (or islands). The hatched region represents a specific accessible subdomain (island). From one accessible island to another, the transport laws are needed for motion continuity, which can be referred to Luo [1]. The passability of flow from the accessible to inaccessible domains will be discussed later also.

Consider a dynamic system consisting of N subdynamic systems in a universal domain $\mho \subset \mathscr{R}^n$. The universal domain is divided into N accessible subdomains $\Omega_\alpha\,(\alpha \in I)$ and the union of inaccessible domain Ω_0. The union of all the accessible subdomains $\cup_{\alpha \in I} \Omega_\alpha$ and $\mho = \cup_{\alpha \in I} \Omega_\alpha \cup \Omega_0$ is the universal domain, as shown

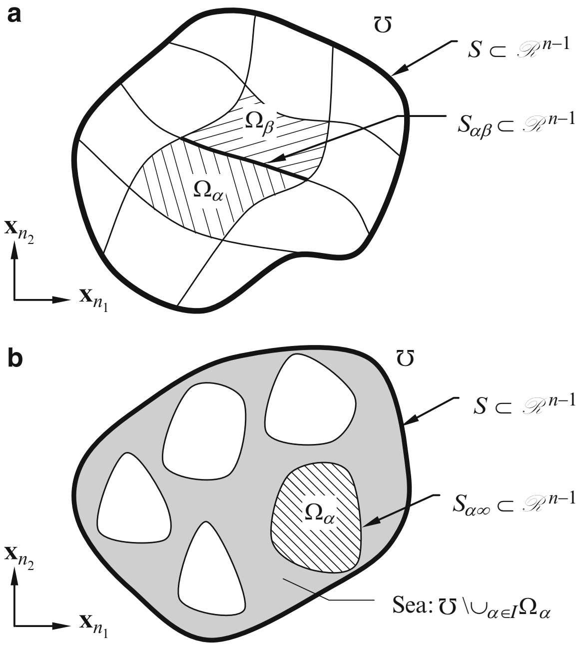

Fig. 2.1 Phase space: (**a**) connectable and (**b**) separable domains ($n_1 + n_2 = n$)

in Fig. 2.1 by an n_1-dimensional, subvector $\mathbf{x}_{n_1}$ and an n_2-dimensional, subvector $\mathbf{x}_{n_2}$ $(n_1 + n_2 = n)$. For the connectable domain in Fig. 2.1a, $\Omega_0 = \varnothing$. In Fig. 2.1b, the union of the inaccessible subdomains is the sea, and $\Omega_0 = \mho \backslash \cup_{\alpha \in I} \Omega_\alpha$ is the complement of the union of the accessible subdomain. On the αth open subdomain Ω_α, there is a C^{r_α}-continuous system ($r_\alpha \geq 1$) in form of

$$\dot{\mathbf{x}}^{(\alpha)} \equiv \mathbf{F}^{(\alpha)}(\mathbf{x}^{(\alpha)}, t, \mathbf{p}_\alpha) \in \mathscr{R}^n,\ \mathbf{x}^{(\alpha)} = (x_1^{(\alpha)}, x_2^{(\alpha)}, \cdots, x_n^{(\alpha)})^{\mathrm{T}} \in \Omega_\alpha. \tag{2.1}$$

The time is t and $\dot{\mathbf{x}}^{(\alpha)} = d\mathbf{x}^{(\alpha)}/dt$. In an accessible subdomain Ω_α, the vector field $\mathbf{F}^{(\alpha)}(\mathbf{x}^{(\alpha)}, t, \mathbf{p}_\alpha)$ with parameter vector $\mathbf{p}_\alpha = (p_\alpha^{(1)}, p_\alpha^{(2)}, \cdots, p_\alpha^{(l)})^{\mathrm{T}} \in \mathscr{R}^l$ is C^{r_α} continuous ($r_\alpha \geq 1$) in $\mathbf{x}^{(\alpha)} \in \Omega_\alpha$ and for all time t; the continuous flow in Eq. (2.1) $\mathbf{x}^{(\alpha)}(t) = \mathbf{\Phi}^{(\alpha)}(\mathbf{x}^{(\alpha)}(t_0), t, \mathbf{p}_\alpha)$ with $\mathbf{x}^{(\alpha)}(t_0) = \mathbf{\Phi}^{(\alpha)}(\mathbf{x}^{(\alpha)}(t_0), t_0, \mathbf{p}_\alpha)$ is $C^{r_\alpha+1}$ continuous for time t.

For discontinuous dynamical systems, the following assumptions will be adopted herein.

H2.1 The flow switching between two adjacent subsystems is time continuous.
H2.2 For an unbounded, accessible subdomain Ω_α, there is a bounded domain $D_\alpha \subset \Omega_\alpha$ and the corresponding vector field and its flow are bounded, i.e.,

$$||\mathbf{F}^{(\alpha)}|| \leq K_1(\text{const}) \text{ and } ||\mathbf{\Phi}^{(\alpha)}|| \leq K_2(\text{const}) \text{ on } D_\alpha \text{ for } t \in [0, \infty). \tag{2.2}$$

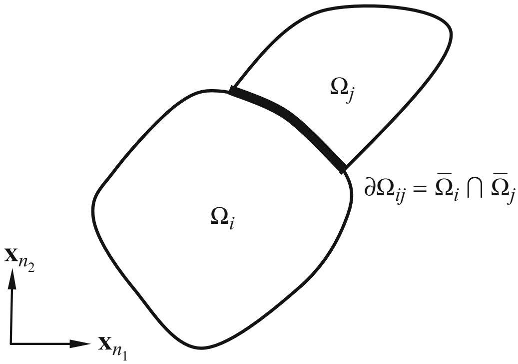

Fig. 2.2 Subdomains Ω_i and Ω_j, and the corresponding boundary $\partial\Omega_{ij}$

H2.3 For a bounded, accessible subdomain Ω_α, there is a bounded domain $D_\alpha \subseteq \Omega_\alpha$ and the corresponding vector field is bounded, but the flow may be unbounded, i.e.,

$$||\mathbf{F}^{(\alpha)}|| \leq K_1(\text{const}) \text{ and } ||\mathbf{\Phi}^{(\alpha)}|| \leq \infty \text{ on } D_\alpha \text{ for } t \in [0, \infty). \tag{2.3}$$

Since dynamical systems on different accessible subdomains are distinguishing, the relation between flows in the two subdomains should be developed herein for flow continuity. For a subdomain Ω_α, there are k_α-adjacent subdomains with k_α-pieces of boundaries. Consider a boundary of any two adjacent subdomains, formed by the intersection of the two closed subdomains (i.e., $\partial\Omega_{ij} = \bar{\Omega}_i \cap \bar{\Omega}_j$) $(i, j \in I, j \neq i)$, as shown in Fig. 2.2.

Definition 2.5 The boundary in n-dimensional phase space is defined as

$$\begin{aligned} S_{ij} &\equiv \partial\Omega_{ij} = \bar{\Omega}_i \cap \bar{\Omega}_j \\ &= \left\{ \mathbf{x} | \varphi_{ij}(\mathbf{x}, t, \boldsymbol{\lambda}) = 0, \varphi_{ij} \text{ is } C^r\text{-continuous } (r \geq 1) \right\} \subset \mathscr{R}^{n-1}. \end{aligned} \tag{2.4}$$

From the definition, $\partial\Omega_{ij} = \partial\Omega_{ji}$. The flow on the boundary $\partial\Omega_{ij}$ can be defined by

$$\dot{\mathbf{x}}^{(0)} = \mathbf{F}^{(0)}(\mathbf{x}^{(0)}, t) \text{ with } \varphi_{ij}(\mathbf{x}^{(0)}, t, \boldsymbol{\lambda}) = 0 \tag{2.5}$$

where $\mathbf{x}^{(0)} = (x_1^{(0)}, x_2^{(0)}, \cdots, x_n^{(0)})^{\mathrm{T}}$. With specific initial conditions, one always obtains different flows on $\varphi_{ij}(\mathbf{x}^{(0)}, t, \boldsymbol{\lambda}) = \varphi_{ij}(\mathbf{x}_0^{(0)}, t_0, \boldsymbol{\lambda}) = 0$.

Definition 2.6 The two subdomains Ω_i and Ω_j are *disjoint* if the boundary $\partial\Omega_{ij}$ is an empty set (i.e., $\partial\Omega_{ij} = \varnothing$).

Definition 2.7 If the intersection of three or more subdomains,

$$\Gamma_{\alpha_1\alpha_2\cdots\alpha_k} \equiv \cap_{\alpha=\alpha_1}^{\alpha_k} \bar{\Omega}_\alpha \subset \mathscr{R}^r, (r = 0, 1, \cdots, n-2) \tag{2.6}$$

where $\alpha_k \in I$ and $k \geq 3$ is nonempty, and the subdomain intersection is termed the *singular* set.

As stated before, a flow $\mathbf{x}_i^{(j)}$ in Ω_i is governed by a dynamical system defined on the jth subdomain Ω_j. This kind of flow is called the *imaginary flow* because the flow is not determined by the dynamical system on its own domain. To further understand dynamical behaviors of discontinuous dynamical systems, it is necessary to introduce imaginary flows. Consider the jth imaginary flow in the ith domain Ω_i is a flow in Ω_i governed by the dynamical system defined on the jth subdomain Ω_j. The two subdomains can be either adjacent or separable. Thus, the mathematical definition of imaginary flows is as follows.

Definition 2.8 The C^{r_j+1} ($r_j \geq 1$)-continuous flow $\mathbf{x}_i^{(j)}(t)$ is termed the jth imaginary flow in the ith open subdomain Ω_i if the flow $\mathbf{x}_i^{(j)}(t)$ is determined by an application of a C^{r_j}-continuous system on the jth open subdomain Ω_j to the ith open subdomain Ω_i, i.e.,

$$\dot{\mathbf{x}}_i^{(j)} = \mathbf{F}^{(j)}(\mathbf{x}_i^{(j)}, t, \mathbf{p}_j) \in \mathscr{R}^n, \mathbf{x}_i^{(j)} = (x_{i1}^{(j)}, x_{i2}^{(j)}, \cdots, x_{in}^{(j)})^{\mathrm{T}} \in \Omega_i, \tag{2.7}$$

with the initial condition $\mathbf{x}_i^{(j)}(t_0) = \mathbf{\Phi}^{(j)}(\mathbf{x}_i^{(j)}(t_0), t_0, \mathbf{p}_j)$.

2.2 G-Functions

Before the general theory for flow passability to a specific boundary in a discontinuous dynamical system is discussed, a concept of G-function will be introduced to measure behaviors of discontinuous dynamical systems in the normal direction of the boundary. The real flow is used to define the G-functions, and such G-functions are also applicable to the imaginary flows. For simplicity, as in Luo [2, 3], consider two infinitesimal time intervals $[t-\varepsilon, t)$ and $(t, t+\varepsilon]$. There are two flows in domain Ω_α ($\alpha = i, j$) and on the boundary $\partial\Omega_{ij}$ determined by Eqs. (2.1) and (2.5), respectively. As in Luo [2, 3], the vector difference between two flows for three time instants is given by $\mathbf{x}_{t-\varepsilon}^{(\alpha)} - \mathbf{x}_{t-\varepsilon}^{(0)}$, $\mathbf{x}_t^{(\alpha)} - \mathbf{x}_t^{(0)}$, and $\mathbf{x}_{t+\varepsilon}^{(\alpha)} - \mathbf{x}_{t+\varepsilon}^{(0)}$. The normal vectors of boundary relative to the corresponding flow $\mathbf{x}^{(0)}(t)$ are expressed by ${}^{t-\varepsilon}\mathbf{n}_{\partial\Omega_{ij}}$, ${}^{t}\mathbf{n}_{\partial\Omega_{ij}}$, and ${}^{t+\varepsilon}\mathbf{n}_{\partial\Omega_{ij}}$ and the corresponding tangential vectors of the flow $\mathbf{x}^{(0)}(t)$ on the boundary are expressed ${}^{t-\varepsilon}\mathbf{t}_{\partial\Omega_{ij}}$, ${}^{t}\mathbf{t}_{\partial\Omega_{ij}}$, and ${}^{t+\varepsilon}\mathbf{t}_{\partial\Omega_{ij}}$, respectively. From the normal vectors of the boundary $\partial\Omega_{ij}$, the dot product functions of the normal vector and the position vector difference between the two flows in domain and on the boundary are defined by

$$\left.\begin{aligned} d_{t-\varepsilon}^{(\alpha)} &= {}^{t-\varepsilon}\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot (\mathbf{x}_{t-\varepsilon}^{(\alpha)} - \mathbf{x}_{t-\varepsilon}^{(0)}), \\ d_{t}^{(\alpha)} &= {}^{t}\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot (\mathbf{x}_{t}^{(\alpha)} - \mathbf{x}_{t}^{(0)}), \\ d_{t+\varepsilon}^{(\alpha)} &= {}^{t+\varepsilon}\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot (\mathbf{x}_{t+\varepsilon}^{(\alpha)} - \mathbf{x}_{t+\varepsilon}^{(0)}) \end{aligned}\right\} \tag{2.8}$$

where the normal vector of the boundary surface $\partial\Omega_{ij}$ at point $\mathbf{x}^{(0)}(t)$ is

$$ {}^{t}\mathbf{n}_{\partial\Omega_{ij}} \equiv \mathbf{n}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}, t, \boldsymbol{\lambda}) = \nabla\varphi_{ij}(\mathbf{x}^{(0)}, t, \boldsymbol{\lambda}) = \left(\frac{\partial\varphi_{ij}}{\partial x_1^{(0)}}, \frac{\partial\varphi_{ij}}{\partial x_2^{(0)}}, \cdots, \frac{\partial\varphi_{ij}}{\partial x_n^{(0)}} \right)^{\mathrm{T}}. \tag{2.9} $$

If ${}^{t}\mathbf{n}_{\partial\Omega_{ij}}$ is a unit vector, the dot product is the normal component which is the distance of the two points of two flows in the normal direction of the boundary surface.

Definition 2.9 Consider a dynamic system in Eq. (2.1) in domain Ω_α $(\alpha \in \{i, j\})$ which has a flow $\mathbf{x}^{(\alpha)} = \boldsymbol{\Phi}(t_0, \mathbf{x}_0^{(\alpha)}, \mathbf{p}_\alpha, t)$ with an initial condition $(t_0, \mathbf{x}_0^{(\alpha)})$, and on the boundary $\partial\Omega_{ij}$, there is an enough smooth flow $\mathbf{x}^{(0)} = \boldsymbol{\Phi}(t_0, \mathbf{x}_0^{(0)}, \boldsymbol{\lambda}, t)$ with an initial condition $(t_0, \mathbf{x}_0^{(0)})$. For an arbitrarily small $\varepsilon > 0$, there are two time intervals $[t-\varepsilon, t)$ or $(t, t+\varepsilon]$ for flow $\mathbf{x}^{(\alpha)}$ $(\alpha \in \{i, j\})$. The G-functions $(G_{\partial\Omega_{ij}}^{(\alpha)})$ of the domain flow $\mathbf{x}^{(\alpha)}$ to the boundary flow $\mathbf{x}^{(0)}$ on the boundary in the normal direction of the boundary $\partial\Omega_{ij}$ are defined as

$$ \begin{aligned} G_{\partial\Omega_{ij}}^{(\alpha)}(\mathbf{x}_t^{(0)}, t_-, \mathbf{x}_{t_-}^{(\alpha)}, \mathbf{p}_\alpha, \boldsymbol{\lambda}) &= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}[{}^{t}\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot (\mathbf{x}_{t_-}^{(\alpha)} - \mathbf{x}_t^{(0)}) - {}^{t-\varepsilon}\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot (\mathbf{x}_{t-\varepsilon}^{(\alpha)} - \mathbf{x}_{t-\varepsilon}^{(0)})], \\ G_{\partial\Omega_{ij}}^{(\alpha)}(\mathbf{x}_t^{(0)}, t_+, \mathbf{x}_{t_+}^{(\alpha)}, \mathbf{p}_\alpha, \boldsymbol{\lambda}) &= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}[{}^{t+\varepsilon}\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot (\mathbf{x}_{t+\varepsilon}^{(\alpha)} - \mathbf{x}_{t+\varepsilon}^{(0)}) - {}^{t}\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot (\mathbf{x}_{t_+}^{(\alpha)} - \mathbf{x}_t^{(0)})]. \end{aligned} \tag{2.10} $$

From Eq. (2.10), since $\mathbf{x}_{t\pm}^{(\alpha)}$ and $\mathbf{x}_t^{(0)}$ are the solutions of Eqs. (2.1) and (2.5), their derivatives exist and $t_{m\pm} \equiv t_m \pm 0$ is to represent the quantity in the domain rather than on the boundary. Further, by use of the Taylor series expansion, Eq. (2.10) gives

$$ G_{\partial\Omega_{ij}}^{(\alpha)}(\mathbf{x}_t^{(0)}, t_{t\pm}, \mathbf{x}_{t\pm}^{(\alpha)}, \mathbf{p}_\alpha, \boldsymbol{\lambda}) = D_0 {}^{t}\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot (\mathbf{x}_{t\pm}^{(\alpha)} - \mathbf{x}_t^{(0)}) + {}^{t}\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot (\dot{\mathbf{x}}_{t\pm}^{(\alpha)} - \dot{\mathbf{x}}_t^{(0)}) \tag{2.11} $$

where the total derivative operators are defined as

$$ D_0(\cdot) \equiv \frac{\partial(\cdot)}{\partial\mathbf{x}^{(0)}}\dot{\mathbf{x}}^{(0)} + \frac{\partial(\cdot)}{\partial t} \text{ and } D_\alpha(\cdot) \equiv \frac{\partial(\cdot)}{\partial\mathbf{x}^{(\alpha)}}\dot{\mathbf{x}}^{(\alpha)} + \frac{\partial(\cdot)}{\partial t}. \tag{2.12} $$

Using Eqs. (2.1) and (2.5), the *G-function* in Eq. (2.11) becomes

$$ \begin{aligned} G_{\partial\Omega_{ij}}^{(\alpha)}(\mathbf{x}_t^{(0)}, t_\pm, \mathbf{x}_{t_\pm}^{(\alpha)}, \mathbf{p}_\alpha, \boldsymbol{\lambda}) = D_0 {}^{t}\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot (\mathbf{x}_{t_\pm}^{(\alpha)} - \mathbf{x}_t^{(0)}) + {}^{t}\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot [\mathbf{F}^{(\alpha)}(\mathbf{x}_{t_\pm}^{(\alpha)}, t_\pm, \mathbf{p}_\alpha) \\ - \mathbf{F}^{(0)}(\mathbf{x}_t^{(0)}, t, \boldsymbol{\lambda})]. \end{aligned} \tag{2.13} $$

Consider the flow contacting with the boundary at time t_m (i.e., $\mathbf{x}_m^{(\alpha)} = \mathbf{x}_m^{(0)}$). Because a flow $\mathbf{x}^{(\alpha)}(t)$ approaches the separation boundary with the zero-order contact (i.e., $\mathbf{x}^{(\alpha)}(t_{m\pm}) = \mathbf{x}_m = \mathbf{x}^{(0)}(t_m)$), the G-function is defined as

$$
\begin{aligned}
&G_{\partial\Omega_{ij}}^{(\alpha)}(\mathbf{x}_t^{(0)}, t_\pm, \mathbf{x}_{t_\pm}^{(\alpha)}, \mathbf{p}_\alpha, \boldsymbol{\lambda}) \\
&= \mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}}(\mathbf{x}^{(0)}, t, \boldsymbol{\lambda}) \cdot [\dot{\mathbf{x}}^{(\alpha)}(t) - \dot{\mathbf{x}}^{(0)}(t)]\big|_{(\mathbf{x}_m^{(0)}, \mathbf{x}_{m\pm}^{(\alpha)}, t_{m\pm})} \\
&= \left[\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}}(\mathbf{x}^{(0)}, t, \boldsymbol{\lambda}) \cdot \dot{\mathbf{x}}^{(\alpha)}(t) + \frac{\partial \varphi_{ij}(\mathbf{x}^{(0)}, t, \boldsymbol{\lambda})}{\partial t} \right] \Bigg|_{(\mathbf{x}_m^{(0)}, \mathbf{x}_{m\pm}^{(\alpha)}, t_{m\pm})} \\
&= \left[\nabla \varphi_{ij}(\mathbf{x}^{(0)}, t, \boldsymbol{\lambda}) \cdot \dot{\mathbf{x}}^{(\alpha)}(t) + \frac{\partial \varphi_{ij}(\mathbf{x}^{(0)}, t, \boldsymbol{\lambda})}{\partial t} \right] \Bigg|_{(\mathbf{x}_m^{(0)}, \mathbf{x}_{m\pm}^{(\alpha)}, t_{m\pm})}
\end{aligned}
\tag{2.14}
$$

With Eqs. (2.1) and (2.5), equation (2.13) can be rewritten as

$$
\begin{aligned}
&G_{\partial\Omega_{ij}}^{(\alpha)}(\mathbf{x}_t^{(0)}, t_\pm, \mathbf{x}_{t_\pm}^{(\alpha)}, \mathbf{p}_\alpha, \boldsymbol{\lambda}) \\
&= \mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}}(\mathbf{x}^{(0)}, t, \boldsymbol{\lambda}) \cdot [\mathbf{F}(\mathbf{x}^{(\alpha)}, t, \mathbf{p}_\alpha) - \mathbf{F}^{(0)}(\mathbf{x}^{(0)}, t, \boldsymbol{\lambda})]\big|_{(\mathbf{x}_m^{(0)}, \mathbf{x}_{m\pm}^{(\alpha)}, t_m\pm)} \\
&= \left[\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}}(\mathbf{x}^{(0)}, t, \boldsymbol{\lambda}) \cdot \mathbf{F}(\mathbf{x}^{(\alpha)}, t, \mathbf{p}_\alpha) + \frac{\partial \varphi_{ij}(\mathbf{x}^{(0)}, t, \boldsymbol{\lambda})}{\partial t} \right] \Bigg|_{(\mathbf{x}_m^{(0)}, \mathbf{x}_{m\pm}^{(\alpha)}, t_m\pm)} \\
&= \left[\nabla \varphi_{ij}(\mathbf{x}^{(0)}, t, \boldsymbol{\lambda}) \cdot \mathbf{F}(\mathbf{x}^{(\alpha)}, t, \mathbf{p}_\alpha) + \frac{\partial \varphi_{ij}(\mathbf{x}^{(0)}, t, \boldsymbol{\lambda})}{\partial t} \right] \Bigg|_{(\mathbf{x}_m^{(0)}, \mathbf{x}_{m\pm}^{(\alpha)}, t_m\pm)}.
\end{aligned}
\tag{2.15}
$$

Note that $G_{\partial\Omega_{ij}}^{(\alpha)}(\mathbf{x}_m, t_{m\pm}, \mathbf{p}_\alpha, \boldsymbol{\lambda})$ is a time rate of the inner product of displacement vector difference and the normal direction $\mathbf{n}_{\partial\Omega_{ij}}(\mathbf{x}_m, t_m, \boldsymbol{\lambda})$. If a flow in a discontinuous system crosses over the boundary $\partial\Omega_{ij}$, $G_{\partial\Omega_{ij}}^{(i)} \neq G_{\partial\Omega_{ij}}^{(j)}$. However, without the boundary, the dynamical system is continuous. Thus, $G_{\partial\Omega_{ij}}^{(i)} = G_{\partial\Omega_{ij}}^{(j)}$. Because the corresponding imaginary flow is the extension of a real flow to the boundary, the real and corresponding imaginary flows are continuous. Therefore, the G-functions to both the real and imaginary flows on the boundary $\partial\Omega_{ij}$ are same.

Definition 2.10 Consider a dynamic system in Eq. (2.1) in domain Ω_α $(\alpha \in \{i, j\})$ which has the flow $\mathbf{x}_t^{(\alpha)} = \boldsymbol{\Phi}(t_0, \mathbf{x}_0^{(\alpha)}, \mathbf{p}_\alpha, t)$ with an initial condition $(t_0, \mathbf{x}_0^{(\alpha)})$, and on the boundary $\partial\Omega_{ij}$, there is an enough smooth flow $\mathbf{x}_t^{(0)} = \boldsymbol{\Phi}(t_0, \mathbf{x}_0^{(0)}, \boldsymbol{\lambda}, t)$ with an initial condition $(t_0, \mathbf{x}_0^{(0)})$. For an arbitrarily small $\varepsilon > 0$, there are two time intervals $[t-\varepsilon, t)$ and $(t, t+\varepsilon]$ for a domain flow $\mathbf{x}_t^{(\alpha)}$ $(\alpha \in \{i, j\})$. The vector fields $\mathbf{F}^{(\alpha)}(\mathbf{x}^{(\alpha)}, t, \mathbf{p}_\alpha)$ and $\mathbf{F}^{(0)}(\mathbf{x}^{(0)}, t, \boldsymbol{\lambda})$ are $C_{[t-\varepsilon, t+\varepsilon]}^{r_\alpha}$-continuous $(r_\alpha \geq k)$ for time t with $||d^{r_\alpha+1}\mathbf{x}_t^{(\alpha)}/dt^{r_\alpha+1}|| < \infty$

and $||d^{r_\alpha+1}\mathbf{x}_t^{(0)}/dt^{r_\alpha+1}||<\infty$. The kth-*order, G-functions* of the domain flow $\mathbf{x}_t^{(\alpha)}$ to the boundary flow $\mathbf{x}_t^{(0)}$ in the normal direction of $\partial\Omega_{ij}$ are defined as

$$
\begin{aligned}
&G_{\partial\Omega_{ij}}^{(k,\alpha)}(\mathbf{x}_t^{(0)},t_-,\mathbf{x}_{t_-}^{(\alpha)},\mathbf{p}_\alpha,\boldsymbol{\lambda})\\
&\quad=(k+1)!\lim_{\varepsilon\to 0}\frac{(-1)^{k+2}}{\varepsilon^{k+1}}[{}^{t}\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}}\cdot(\mathbf{x}_{t_-}^{(\alpha)}-\mathbf{x}_t^{(0)})-{}^{t-\varepsilon}\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}}\cdot(\mathbf{x}_{t-\varepsilon}^{(\alpha)}-\mathbf{x}_{t-\varepsilon}^{(0)})\\
&\qquad+\sum\nolimits_{s=0}^{k-1}\frac{1}{(s+1)!}G_{\partial\Omega_{ij}}^{(s,\alpha)}(\mathbf{x}_t^{(0)},t,\mathbf{x}_{t_-}^{(\alpha)},\mathbf{p}_\alpha,\boldsymbol{\lambda})(-\varepsilon)^{s+1}],\\
&G_{\partial\Omega_{ij}}^{(k,\alpha)}(\mathbf{x}_t^{(0)},t_+,\mathbf{x}_{t_+}^{(\alpha)},\mathbf{p}_\alpha,\boldsymbol{\lambda})\\
&\quad=(k+1)!\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^{k+1}}[{}^{t+\varepsilon}\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}}\cdot(\mathbf{x}_{t+\varepsilon}^{(\alpha)}-\mathbf{x}_{t+\varepsilon}^{(0)})-{}^{t}\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}}\cdot(\mathbf{x}_{t_+}^{(\alpha)}-\mathbf{x}_t^{(0)})\\
&\qquad-\sum\nolimits_{s=0}^{k-1}\frac{1}{(s+1)!}G_{\partial\Omega_{ij}}^{(s,\alpha)}(\mathbf{x}_t^{(0)},t,\mathbf{x}_{t_+}^{(\alpha)},\mathbf{p}_\alpha,\boldsymbol{\lambda})\varepsilon^{s+1}].
\end{aligned}
\tag{2.16}
$$

Again, the Taylor series expansion applying to Eq. (2.16) yields

$$
\begin{aligned}
&G_{\partial\Omega_{ij}}^{(k,\alpha)}(\mathbf{x}_t^{(0)},t_\pm,\mathbf{x}_{t_\pm}^{(\alpha)},\mathbf{p}_\alpha,\boldsymbol{\lambda})\\
&\quad=\sum\nolimits_{s=0}^{k+1}C_{k+1}^{s}D_0^{k+1-s}\,{}^{t}\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}}\cdot\left(\frac{d^s\mathbf{x}^{(\alpha)}}{dt^s}-\frac{d^s\mathbf{x}^{(0)}}{dt^s}\right)\Big|_{(\mathbf{x}_t^{(0)},\mathbf{x}_{t_\pm}^{(\alpha)},t_\pm)}.
\end{aligned}
\tag{2.17}
$$

Using Eqs. (2.1) and (2.5), the kth-order G-function of the flow $\mathbf{x}_t^{(\alpha)}$ to the boundary $\partial\Omega_{ij}$ is computed by

$$
\begin{aligned}
&G_{\partial\Omega_{ij}}^{(k,\alpha)}(\mathbf{x}_t^{(0)},t_+,\mathbf{x}_{t_+}^{(\alpha)},\mathbf{p}_\alpha,\boldsymbol{\lambda})\\
&\quad=\sum\nolimits_{s=1}^{k+1}C_{k+1}^{s}D_0^{k+1-s}\,{}^{t}\mathbf{n}_{\partial\Omega_{ij}}^{T}\cdot\Big[D_\alpha^{s-1}\mathbf{F}^{(\alpha)}\big(\mathbf{x}^{(\alpha)},t,\mathbf{p}_\alpha\big)\\
&\qquad-D_0^{s-1}\mathbf{F}^{(0)}\big(\mathbf{x}^{(0)},t,\boldsymbol{\lambda}\big)\Big]\Big|_{(\mathbf{x}_t^{(0)},\mathbf{x}_{t_\pm}^{(\alpha)},t_\pm)}+D_0^{k+1}\,{}^{t}\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}}\cdot\big(\mathbf{x}_{t_\pm}^{(\alpha)}-\mathbf{x}_t^{(0)}\big),
\end{aligned}
\tag{2.18}
$$

where

$$
C_{k+1}^{s}=\frac{(k+1)!}{s!(k+1-s)!}
\tag{2.19}
$$

with $C_{k+1}^{0}=1$ and $s!=1\times 2\times\cdots\times s$.

The G-function $G_{\partial\Omega_{ij}}^{(k,\alpha)}$ is the time rate of $G_{\partial\Omega_{ij}}^{(k-1,\alpha)}$. If a flow contacting with $\partial\Omega_{ij}$ at time t_m (i.e., $\mathbf{x}_{m\pm}^{(\alpha)}=\mathbf{x}_m^{(0)}$) and ${}^{t}\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}}\equiv\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}}$, the kth-order G-function is

$$\begin{aligned} &G_{\partial\Omega_{ij}}^{(k,\alpha)}(\mathbf{x}_m, t_{m\pm}, \mathbf{p}_\alpha, \boldsymbol{\lambda}) \\ &= \sum\nolimits_{r=1}^{k+1} C_{k+1}^r D_0^{k+1-r} \mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \left[\frac{d^r \mathbf{x}^{(\alpha)}}{dt^r} - \frac{d^r \mathbf{x}^{(0)}}{dt^r} \right] \Bigg|_{(\mathbf{x}_m^{(0)}, \mathbf{x}_{m\pm}^{(\alpha)}, t_{m\pm})} \\ &= \sum\nolimits_{r=1}^{k+1} C_{k+1}^r D_0^{k+1-r} \mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot [D_\alpha^{r-1} \mathbf{F}(\mathbf{x}^{(\alpha)}, t, \mathbf{p}_\alpha) \\ &\quad - D_0^{r-1} \mathbf{F}^{(0)}(\mathbf{x}^{(0)}, t, \boldsymbol{\lambda})]|_{(\mathbf{x}_m^{(0)}, \mathbf{x}_{m\pm}^{(\alpha)}, t_{m\pm})} \end{aligned} \tag{2.20}$$

For $k = 0$, one obtains

$$G_{\partial\Omega_{ij}}^{(k,\alpha)}(\mathbf{x}_m, t_{m\pm}, \mathbf{p}_\alpha, \boldsymbol{\lambda}) = G_{\partial\Omega_{ij}}^{(\alpha)}(\mathbf{x}_m, t_{m\pm}, \mathbf{p}_\alpha, \boldsymbol{\lambda}). \tag{2.21}$$

From now on, $\mathbf{n}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}) \equiv \mathbf{n}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}, t, \boldsymbol{\lambda})$.

2.3 Passable Flows

Compared to the continuous dynamical systems, discontinuous dynamical systems possess many passable flows to the boundary $\partial\Omega_{ij}$ because $G_{\partial\Omega_{ij}}^{(i)} \neq G_{\partial\Omega_{ij}}^{(j)}$. A passable flow to a specific boundary is discussed first, as sketched in Fig. 2.3. $\mathbf{x}^{(i)}(t)$ and $\mathbf{x}^{(j)}(t)$ represent the *real* flows in domains Ω_i and Ω_j, respectively. They are depicted by thin solid curves. $\mathbf{x}_i^{(j)}(t)$ and $\mathbf{x}_j^{(i)}(t)$ are the *imaginary* flows in domains Ω_i and Ω_j, respectively, controlled by the vector fields on Ω_j and Ω_i. Such imaginary flows are depicted by dashed curves. The hollow circles are switching points, and filled circles are starting points. The detail discussion of the real and imaginary flows can be found from Luo [4, 5]. The flow on the boundary is described by $\mathbf{x}^{(0)}(t)$. The normal and tangential vectors $\mathbf{n}_{\partial\Omega_{ij}}$ and $\mathbf{t}_{\partial\Omega_{ij}}$ on the boundary are depicted. The passable flow to a specific boundary is defined as follows.

Definition 2.11 For a discontinuous dynamical system in Eq. (2.1), there is a point $\mathbf{x}^{(0)}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ at time t_m between two adjacent domains Ω_α ($\alpha = i, j$). For an arbitrarily small $\varepsilon > 0$, there are two time intervals $[t_{m-\varepsilon}, t_m)$ and $(t_m, t_{m+\varepsilon}]$. Suppose $\mathbf{x}^{(i)}(t_{m-}) = \mathbf{x}_m = \mathbf{x}^{(j)}(t_{m+})$. The flow $\mathbf{x}^{(i)}(t)$ and $\mathbf{x}^{(j)}(t)$ to the boundary $\partial\Omega_{ij}$ is *semi-passable* from domain Ω_i to Ω_j if

$$\left.\begin{array}{ll} \text{either} & \left.\begin{array}{l} \mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}}(\mathbf{x}_{m-\varepsilon}^{(0)}) \cdot [\mathbf{x}_{m-\varepsilon}^{(0)} - \mathbf{x}_{m-\varepsilon}^{(i)}] > 0 \\ \mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}}(\mathbf{x}_{m+\varepsilon}^{(0)}) \cdot [\mathbf{x}_{m+\varepsilon}^{(j)} - \mathbf{x}_{m+\varepsilon}^{(0)}] > 0 \end{array}\right\} \text{for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_j \\ \text{or} & \left.\begin{array}{l} \mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}}(\mathbf{x}_{m-\varepsilon}^{(0)}) \cdot [\mathbf{x}_{m-\varepsilon}^{(0)} - \mathbf{x}_{m-\varepsilon}^{(i)}] < 0 \\ \mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}}(\mathbf{x}_{m+\varepsilon}^{(0)}) \cdot [\mathbf{x}_{m+\varepsilon}^{(j)} - \mathbf{x}_{m+\varepsilon}^{(0)}] < 0 \end{array}\right\} \text{for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_i. \end{array}\right\} \tag{2.22}$$

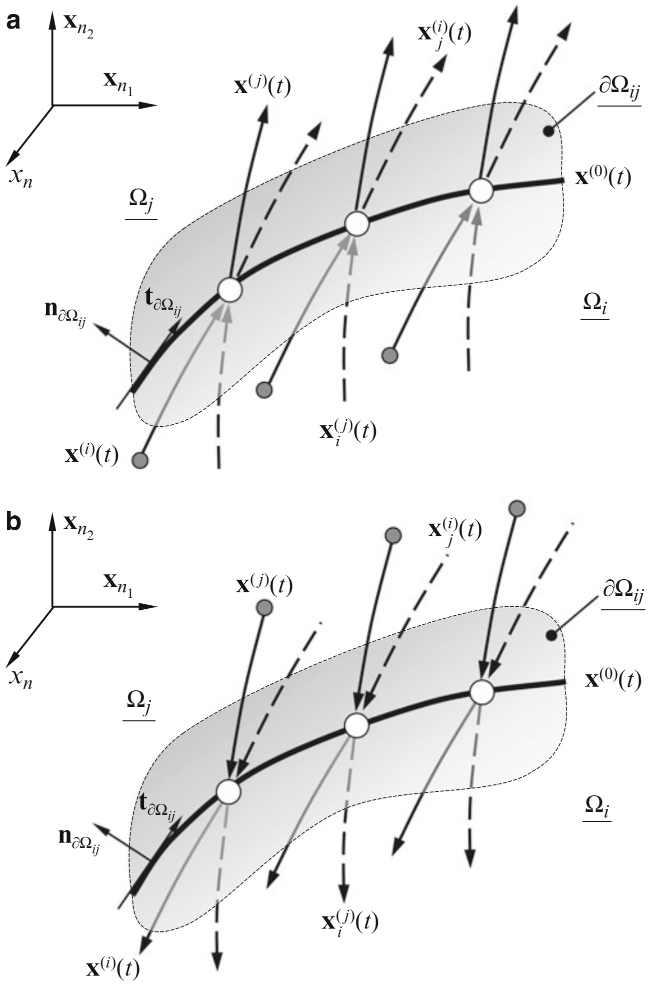

Fig. 2.3 Passable flows: (**a**) from Ω_i to Ω_j with the $(2k_i : 2k_j)$-order and (**b**) from Ω_j to Ω_i with the $(2k_j : 2k_i)$-order. *Real* flows $\mathbf{x}^{(i)}(t)$ and $\mathbf{x}^{(j)}(t)$ in domains Ω_i and Ω_j are depicted by *thin solid curves*, respectively. *Imaginary* flows $\mathbf{x}_i^{(j)}(t)$ and $\mathbf{x}_j^{(i)}(t)$ in domains Ω_i and Ω_j, which are defined by the vector fields in Ω_j and Ω_i are depicted by *dashed curves*, respectively. The flow on the boundary is described by $\mathbf{x}^{(0)}(t)$. The normal and tangential vectors $\mathbf{n}_{\partial\Omega_{ij}}$ and $\mathbf{t}_{\partial\Omega_{ij}}$ of the boundary are depicted. *Hollow circles* are for switching points on the boundary and *filled circles* are for starting points $(n_1 + n_2 + 1 = n)$

Since flow properties in domains Ω_i and Ω_j are different at point $(t_m, \mathbf{x}_m)$, $G^{(i)}_{\partial\Omega_{ij}} \neq G^{(j)}_{\partial\Omega_{ij}}$ to $\partial\Omega_{ij}$. The necessary and sufficient conditions for such a passable flow on $\partial\Omega_{ij}$ from domain Ω_i to Ω_j are given as follows.

Theorem 2.1 *For a discontinuous dynamical system in Eq. (2.1), there is a point* $\mathbf{x}^{(0)}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ *at time* t_m *between two adjacent domains* Ω_α $(\alpha = i, j)$*. For an* *arbitrarily small* $\varepsilon > 0$*, there are two time intervals* $[t_{m-\varepsilon}, t_m)$ *and* $(t_m, t_{m+\varepsilon}]$*. Suppose* $\mathbf{x}^{(i)}(t_{m-}) = \mathbf{x}_m = \mathbf{x}^{(j)}(t_{m+})$*. Two flows* $\mathbf{x}^{(i)}(t)$ *and* $\mathbf{x}^{(j)}(t)$ *are* $C^{r_i}_{[t_{m-\varepsilon}, t_m)}$ *and* $C^{r_j}_{(t_m, t_{m+\varepsilon}]}$*-continuous for time t, respectively, and* $\|d^{r_\alpha+1}\mathbf{x}^{(\alpha)}/dt^{r_\alpha+1}\| < \infty$ $(r_\alpha \geq 1,\ \alpha = i,j)$*. The flow* $\mathbf{x}^{(i)}(t)$ *and* $\mathbf{x}^{(j)}(t)$ *to the boundary* $\partial\Omega_{ij}$ *is semi-passable from domain* Ω_i *to* Ω_j *if and only if*

$$\begin{aligned} &\text{either} \quad \left.\begin{aligned} G^{(i)}_{\partial\Omega_{ij}}(\mathbf{x}_m, t_{m-}, \mathbf{p}_i, \boldsymbol{\lambda}) > 0 \\ G^{(j)}_{\partial\Omega_{ij}}(\mathbf{x}_m, t_{m+}, \mathbf{p}_j, \boldsymbol{\lambda}) > 0 \end{aligned}\right\} \text{ for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_j, \\ &\text{or} \quad \left.\begin{aligned} G^{(i)}_{\partial\Omega_{ij}}(\mathbf{x}_m, t_{m-}, \mathbf{p}_i, \boldsymbol{\lambda}) < 0 \\ G^{(j)}_{\partial\Omega_{ij}}(\mathbf{x}_m, t_{m+}, \mathbf{p}_j, \boldsymbol{\lambda}) < 0 \end{aligned}\right\} \text{ for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_i. \end{aligned} \tag{2.23}$$

Proof For a point $\mathbf{x}_m \in \partial\Omega_{ij}$, suppose $\mathbf{x}^{(i)}(t_{m-}) = \mathbf{x}_m = \mathbf{x}^{(j)}(t_{m+})$. Two flows $\mathbf{x}^{(i)}(t)$ and $\mathbf{x}^{(j)}(t)$ are $C^{r_i}_{[t_{m-\varepsilon}, t_m)}$ and $C^{r_j}_{(t_m, t_{m+\varepsilon}]}$-continuous ($r_\alpha \geq 1$, $\alpha = i,j$) for time t. $\|\ddot{\mathbf{x}}^{(\alpha)}(t)\| < \infty$ for $0 < \varepsilon << 1$. For $a \in [t_{m-\varepsilon}, t_{m-})$ or $a \in (t_{m+}, t_{m+\varepsilon}]$, the Taylor series expansions of $\mathbf{x}^{(\alpha)}(t_{m\pm\varepsilon})$ with $t_{m\pm\varepsilon} = t_m \pm \varepsilon$ ($\alpha \in \{i,j\}$) to $\mathbf{x}^{(\alpha)}(a)$ give

$$\mathbf{x}^{(\alpha)}_{m\pm\varepsilon} \equiv \mathbf{x}^{(\alpha)}(t_{m\pm} \pm \varepsilon) = \mathbf{x}^{(\alpha)}(a) + \dot{\mathbf{x}}^{(\alpha)}(a)(t_{m\pm} \pm \varepsilon - a) + o(t_{m\pm} \pm \varepsilon - a),$$

As $a \to t_{m\pm}$, taking the limit of the foregoing equation leads to

$$\mathbf{x}^{(\alpha)}_{m\pm\varepsilon} \equiv \mathbf{x}^{(\alpha)}(t_{m\pm} \pm \varepsilon) = \mathbf{x}^{(\alpha)}(t_{m\pm}) + \dot{\mathbf{x}}^{(\alpha)}(t_{m\pm})(\pm\varepsilon) + o(\pm\varepsilon),$$

With Eq. (2.1), one obtains

$$\mathbf{x}^{(\alpha)}_{m\pm\varepsilon} = \mathbf{x}^{(\alpha)}(t_{m\pm}) + \mathbf{F}^{(\alpha)}(t_{m\pm})(\pm\varepsilon) + o(\pm\varepsilon),$$

In a similar fashion, the flow on the boundary is expressed by

$$\begin{aligned} \mathbf{x}^{(0)}_{m\pm\varepsilon} &\equiv \mathbf{x}^{(0)}(t_{m\pm} \pm \varepsilon) = \mathbf{x}^{(0)}(t_{m\pm}) + \dot{\mathbf{x}}^{(0)}(t_{m\pm})(\pm\varepsilon) + o(\pm\varepsilon), \\ &= \mathbf{x}^{(0)}(t_{m\pm}) + \mathbf{F}^{(0)}(t_{m\pm})(\pm\varepsilon) + o(\pm\varepsilon), \end{aligned}$$

$$\mathbf{n}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}_m) = \mathbf{n}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}_m) + D_0\mathbf{n}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}_m)(\pm\varepsilon) + o(\pm\varepsilon).$$

The ignorance of the ε^2-term and high order terms, the deformation of the above equation and left multiplication of $\mathbf{n}_{\partial\Omega_{ij}}$ gives

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}_{m-\varepsilon}) \cdot [\mathbf{x}^{(i)}_{m-\varepsilon} - \mathbf{x}^{(0)}_{m-\varepsilon}] = \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}_m) \cdot [\mathbf{x}^{(i)}_{m-} - \mathbf{x}^{(0)}_m] - \varepsilon G^{(i)}_{\partial\Omega_{ij}}(\mathbf{x}_m, t_m, \mathbf{p}_i, \boldsymbol{\lambda}),$$

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}_{m+\varepsilon}) \cdot [\mathbf{x}^{(j)}_{m+\varepsilon} - \mathbf{x}^{(0)}_{m+\varepsilon}] = \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}_m) \cdot [\mathbf{x}^{(j)}_{m+} - \mathbf{x}^{(0)}_m] + \varepsilon G^{(j)}_{\partial\Omega_{ij}}(\mathbf{x}_m, t_m, \mathbf{p}_j, \boldsymbol{\lambda}).$$

Because of $\mathbf{x}_{m\pm}^{(\alpha)} = \mathbf{x}_m^{(0)} = \mathbf{x}_m$, the foregoing equation becomes

$$\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}}(\mathbf{x}_{m-\varepsilon}^{(0)}) \cdot [\mathbf{x}_{m-\varepsilon}^{(0)} - \mathbf{x}_{m-\varepsilon}^{(i)}] = \varepsilon G_{\partial\Omega_{ij}}^{(i)}(\mathbf{x}_m, t_{m-}, \mathbf{p}_i, \boldsymbol{\lambda}),$$
$$\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}}(\mathbf{x}_{m+\varepsilon}^{(0)}) \cdot [\mathbf{x}_{m+\varepsilon}^{(j)} - \mathbf{x}_{m+\varepsilon}^{(0)}] = \varepsilon G_{\partial\Omega_{ij}}^{(j)}(\mathbf{x}_m, t_{m+}, \mathbf{p}_j, \boldsymbol{\lambda}).$$

With Eq. (2.23), the foregoing equation gives Eq. (2.22). Using Eq. (2.22), Eq. (2.23) can be obtained. This theorem is proved. □

In general, the G-function in Section 2.2 is used to describe the $(2k_i : 2k_j)$-semi-passable flow and the $(2k_i : 2k_j - 1)$-semi-passable flow to the boundary. Without any switching law or transport law on the boundary, the two semi-passable flow can be described by the $(2k_i : m_j)$-semi-passable flow ($k_i, m_j \in \mathbf{N}$). However, in Luo [5, 6], the semi-passable flow with the higher order singularity to the boundary was discussed only for either the *plane* boundary surface or the higher order contact of the flow and the boundary surface.

Definition 2.12 For a discontinuous dynamical system in Eq. (2.1), there is a point $\mathbf{x}^{(0)}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ at time t_m between two adjacent domains Ω_α ($\alpha = i, j$). For an arbitrarily small $\varepsilon > 0$, there are two time intervals $[t_{m-\varepsilon}, t_m)$ and $(t_m, t_{m+\varepsilon}]$. Suppose $\mathbf{x}^{(i)}(t_{m-}) = \mathbf{x}_m = \mathbf{x}^{(j)}(t_{m+})$. A flow $\mathbf{x}^{(i)}(t)$ is $C_{[t_{m-\varepsilon}, t_m)}^{r_i}$ continuous for time t with $||d^{r_i+1}\mathbf{x}^{(j)}/dt^{r_i+1}|| < \infty$ ($r_i \geq 2k_i + 1$), and a flow $\mathbf{x}^{(j)}(t)$ is $C_{(t_m, t_{m+\varepsilon}]}^{r_j}$-continuous with $||d^{r_j+1}\mathbf{x}^{(j)}/dt^{r_j+1}|| < \infty$ ($r_j \geq m_j + 1$). The flow $\mathbf{x}^{(i)}(t)$ of the $(2k_i)$th-order and $\mathbf{x}^{(j)}(t)$ of the m_jth-order to the boundary $\partial\Omega_{ij}$ is $(2k_i : m_j)$-*semi-passable* from domain Ω_i to Ω_j if

$$\left.\begin{aligned} &G_{\partial\Omega_{ij}}^{(s,i)}(\mathbf{x}_m, t_{m-}, \mathbf{p}_i, \boldsymbol{\lambda}) = 0 \text{ for } s = 0, 1, \cdots, 2k_i - 1 \\ &G_{\partial\Omega_{ij}}^{(2k_i,i)}(\mathbf{x}_m, t_{m-}, \mathbf{p}_i, \boldsymbol{\lambda}) \neq 0, \end{aligned}\right\} \tag{2.24}$$

$$\left.\begin{aligned} &G_{\partial\Omega_{ij}}^{(s,j)}(\mathbf{x}_m, t_{m+}, \mathbf{p}_j, \boldsymbol{\lambda}) = 0 \text{ for } s = 0, 1, \cdots, m_j - 1 \\ &G_{\partial\Omega_{ij}}^{(m_j,j)}(\mathbf{x}_m, t_{m+}, \mathbf{p}_j, \boldsymbol{\lambda}) \neq 0, \end{aligned}\right\} \tag{2.25}$$

$$\left.\begin{aligned} &\text{either} \left.\begin{aligned} &\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}}(\mathbf{x}_{m-\varepsilon}^{(0)}) \cdot [\mathbf{x}_{m-\varepsilon}^{(0)} - \mathbf{x}_{m-\varepsilon}^{(i)}] > 0 \\ &\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}}(\mathbf{x}_{m+\varepsilon}^{(0)}) \cdot [\mathbf{x}_{m+\varepsilon}^{(j)} - \mathbf{x}_{m+\varepsilon}^{(0)}] > 0 \end{aligned}\right\} \text{for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_j \\ &\text{or} \left.\begin{aligned} &\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}}(\mathbf{x}_{m-\varepsilon}^{(0)}) \cdot [\mathbf{x}_{m-\varepsilon}^{(0)} - \mathbf{x}_{m-\varepsilon}^{(i)}] < 0 \\ &\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}}(\mathbf{x}_{m+\varepsilon}^{(0)}) \cdot [\mathbf{x}_{m+\varepsilon}^{(j)} - \mathbf{x}_{m+\varepsilon}^{(0)}] < 0 \end{aligned}\right\} \text{for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_i. \end{aligned}\right\} \tag{2.26}$$

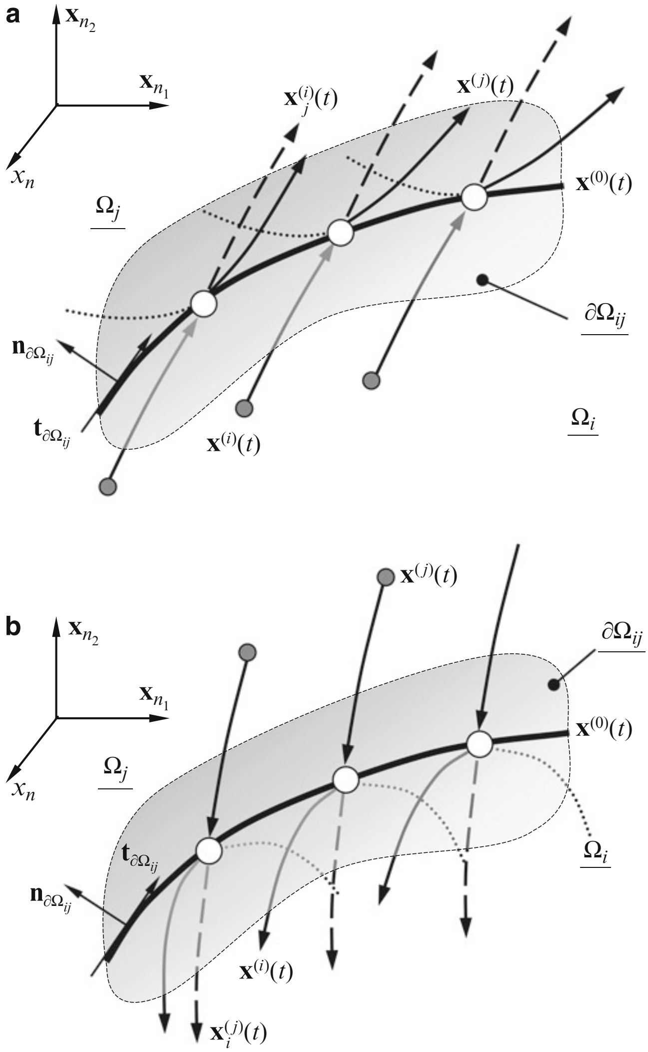

Fig. 2.4 Passable flows: (**a**) from Ω_i to Ω_j with $(2k_i : 2k_j - 1)$-order and (**b**) from Ω_j to Ω_i with $(2k_j : 2k_j - 1)$-order. *Real* flows $\mathbf{x}^{(i)}(t)$ and $\mathbf{x}^{(j)}(t)$ in Ω_i and Ω_j are depicted by *thin solid curves*, respectively. *Imaginary* flows $\mathbf{x}_i^{(j)}(t)$ and $\mathbf{x}_j^{(i)}(t)$ in Ω_i and Ω_j, which are defined by vector fields in Ω_j and Ω_i, are depicted by the *dashed curves*, respectively. The flow on the boundary is described by $\mathbf{x}^{(0)}(t)$. The normal and tangential vectors $\mathbf{n}_{\partial\Omega_{ij}}$ and $\mathbf{t}_{\partial\Omega_{ij}}$ of the boundary are depicted. *Dotted curves* represent tangential flows before time t_{m+}. *Hollow circles* are for switching points on the boundary, and *filled circles* are for starting points $(n_1 + n_2 + 1 = n)$

If $m_j = 2k_j$, the $(2k_i : 2k_j)$-passable flow can be sketched as in Fig. 2.3. However, for $m_j = 2k_j - 1$, the $(2k_i : 2k_j - 1)$-passable flow from domain Ω_i to Ω_j is sketched in Fig. 2.4a. The tangential flow of the $(2k_j - 1)$th-order exists in domain Ω_j.

The dotted curves represent the tangential curves to the boundary for time $t \in [t_{m-\varepsilon}, t_m)$. The starting point of the flow is $(t_{m-\varepsilon}, \mathbf{x}^{(i)}_{m-\varepsilon})$ in domain Ω_i. If the flow arrives to the point $(t_m, \mathbf{x}_m)$ of the boundary $\partial\Omega_{ij}$, the flow will follow the tangential flow in domain Ω_j. The $(2k_j : 2k_i - 1)$-passable flow from domain Ω_j to Ω_i is presented in Fig. 2.4b with the same behavior as in Fig. 2.4a. So, a new semi-passable flow is formed as the post-transversal, tangential flow discussed in Luo [5, 6]. From the definition of the $(2k_i : m_j)$-passable flow, the corresponding necessary and sufficient conditions can be given by the following theorem.

Theorem 2.2 *For a discontinuous dynamical system in Eq. (2.1), there is a point $\mathbf{x}^{(0)}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ at time t_m between two adjacent domains Ω_α $(\alpha = i, j)$. For an arbitrarily small $\varepsilon > 0$, there are two time intervals $[t_{m-\varepsilon}, t_m)$ and $(t_m, t_{m+\varepsilon}]$. Suppose $\mathbf{x}^{(i)}(t_{m-}) = \mathbf{x}_m = \mathbf{x}^{(j)}(t_{m+})$. A flow $\mathbf{x}^{(i)}(t)$ is $C^r_{[t_{m-\varepsilon}, t_m)}$ continuous for time t with $\|d^{r_i+1}\mathbf{x}^{(i)}/dt^{r_i+1}\| < \infty$ $(r_i \geq 2k_i + 1)$, and a flow $\mathbf{x}^{(j)}(t)$ is $C^{r_j}_{(t_m, t_{m+\varepsilon}]}$-continuous with $\|d^{r_j+1}\mathbf{x}^{(j)}/dt^{r_j+1}\| < \infty$ $(r_j \geq m_j + 1)$. The flow $\mathbf{x}^{(i)}(t)$ of the $(2k_i)$th-order and $\mathbf{x}^{(j)}(t)$ of the m_jth-order to the boundary $\partial\Omega_{ij}$ is $(2k_i : m_j)$-semi-passable from domain Ω_i to Ω_j if and only if*

$$G^{(s,i)}_{\partial\Omega_{ij}}(\mathbf{x}_m, t_{m-}, \mathbf{p}_i, \boldsymbol{\lambda}) = 0 \text{ for } s = 0, 1, \cdots, 2k_i - 1; \tag{2.27}$$

$$G^{(s,j)}_{\partial\Omega_{ij}}(\mathbf{x}_m, t_{m+}, \mathbf{p}_j, \boldsymbol{\lambda}) = 0 \text{ for } s = 0, 1, \cdots, m_j - 1; \tag{2.28}$$

$$\left.\begin{array}{ll} \text{either} & \left.\begin{array}{l} G^{(2k_i,i)}_{\partial\Omega_{ij}}(\mathbf{x}_m, t_{m-}, \mathbf{p}_i, \boldsymbol{\lambda}) > 0 \\ G^{(m_j,j)}_{\partial\Omega_{ij}}(\mathbf{x}_m, t_{m+}, \mathbf{p}_j, \boldsymbol{\lambda}) > 0 \end{array}\right\} \text{ for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_j \\ \text{or} & \left.\begin{array}{l} G^{(2k_i,i)}_{\partial\Omega_{ij}}(\mathbf{x}_m, t_{m-}, \mathbf{p}_i, \boldsymbol{\lambda}) < 0 \\ G^{(m_j,j)}_{\partial\Omega_{ij}}(\mathbf{x}_m, t_{m+}, \mathbf{p}_j, \boldsymbol{\lambda}) < 0 \end{array}\right\} \text{ for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_i. \end{array}\right\} \tag{2.29}$$

Proof For a point $\mathbf{x}_m \in \partial\Omega_{ij}$, suppose $\mathbf{x}^{(i)}(t_{m-}) = \mathbf{x}_m = \mathbf{x}^{(j)}(t_{m+})$. The flow $\mathbf{x}^{(i)}(t)$ is $C^{r_i}_{[t_{m-\varepsilon}, t_m)}$-continuous $(r_i \geq 2k_i + 1)$ for time t and $\|d^{r_i+1}\mathbf{x}^{(i)}/dt^{r_i+1}\| < \infty$. The flow $\mathbf{x}^{(j)}(t)$ is $C^{r_j}_{(t_m, t_{m+\varepsilon}]}$-continuous for time t, and $\|d^{r_j+1}\mathbf{x}^{(j)}/dt^{r_j+1}\| < \infty$ $(r_j \geq m_j + 1)$. Equations (2.27) and (2.28) are identical to the first equations of Eqs. (2.24) and (2.25), respectively. Equation (2.29) implies the second equations of Eqs. (2.24) and (2.25). For $a \in [t_{m-\varepsilon}, t_{m-})$ or $b \in (t_{m+}, t_{m+\varepsilon}]$, the Taylor series of $\mathbf{x}^{(i)}(t_{m-\varepsilon})$ and $\mathbf{x}^{(j)}(t_{m+\varepsilon})$ at $\mathbf{x}^{(i)}(a)$ and $\mathbf{x}^{(j)}(b)$ up to the ε^{2k_i+1} and ε^{m_j+1}-terms give

$$\begin{aligned}
\mathbf{x}^{(i)}_{m-\varepsilon} &\equiv \mathbf{x}^{(i)}(t_{m-} - \varepsilon) \\
&= \mathbf{x}^{(i)}(a) + \sum\nolimits_{s=1}^{2k_i} \frac{1}{s!} \left.\frac{d^s\mathbf{x}^{(i)}}{dt^s}\right|_{t=a} (t_{m-} - \varepsilon - a)^s \\
&\quad + \frac{1}{(2k_i+1)!} \left.\frac{d^{2k_i+1}\mathbf{x}^{(i)}}{dt^{2k_i+1}}\right|_{t=a} (t_{m-} - \varepsilon - a)^{2k_i+1} + o((t_{m-} - \varepsilon - a)^{2k_i+1}),
\end{aligned}$$

$$\begin{aligned}\mathbf{x}_{m+\varepsilon}^{(j)} &\equiv \mathbf{x}^{(j)}(t_{m+}+\varepsilon)\\ &= \mathbf{x}^{(j)}(b) + \sum\nolimits_{s=1}^{m_j} \frac{1}{s!} \frac{d^s \mathbf{x}^{(j)}}{dt^s}\bigg|_{t=b} (t_{m+}+\varepsilon-b)^s\\ &\quad + \frac{1}{m_j!} \frac{d^{m_j+1} \mathbf{x}^{(j)}}{dt^{m_j+1}}\bigg|_{t=b} (t_{m+}+\varepsilon-b)^{m_j+1} + o((t_{m+}+\varepsilon-b)^{m_j+1}),\end{aligned}$$

As $a \to t_{m-}$ and $b \to t_{m+}$, taking the limit of the foregoing equations leads to

$$\begin{aligned}\mathbf{x}_{m-\varepsilon}^{(i)} &\equiv \mathbf{x}^{(i)}(t_{m-}-\varepsilon)\\ &= \mathbf{x}^{(i)}(t_{m-}) + \sum\nolimits_{s=1}^{2k_i} \frac{1}{s!} \frac{d^s \mathbf{x}^{(i)}}{dt^s}\bigg|_{t=t_{m-}} (-\varepsilon)^s + \frac{1}{(2k_i+1)!} \frac{d^{2k_i+1} \mathbf{x}^{(i)}}{dt^{2k_i+1}}\bigg|_{t=t_{m-}} (-\varepsilon)^{2k_i+1}\\ &\quad + o((-\varepsilon)^{2k_i+1})\\ &= \mathbf{x}^{(i)}(t_{m-}) + \sum\nolimits_{s=1}^{2k_i} \frac{1}{s!} D_i^{s-1}\mathbf{F}^{(i)}(t_{m-})(-\varepsilon)^s + \frac{1}{(2k_i+1)!} D_i^{2k_i}\mathbf{F}^{(i)}(t_{m-})(-\varepsilon)^{2k_i+1}\\ &\quad + o((-\varepsilon)^{2k_i+1}),\end{aligned}$$

$$\begin{aligned}\mathbf{x}_{m+\varepsilon}^{(j)} &\equiv \mathbf{x}^{(j)}(t_{m+}+\varepsilon)\\ &= \mathbf{x}^{(j)}(t_{m+}) + \sum\nolimits_{s=1}^{m_j} \frac{1}{m_j!} \frac{d^s \mathbf{x}^{(j)}}{dt^s}\bigg|_{t=t_{m+}} \varepsilon^s + \frac{1}{(m_j+1)!} \frac{d^{m_j+1} \mathbf{x}^{(i)}}{dt^{m_j+1}}\bigg|_{t=t_{m+}} \varepsilon^{m_j+1} + o(\varepsilon^{m_j+1})\\ &= \mathbf{x}^{(j)}(t_{m+}) + \sum\nolimits_{s=1}^{m_j} \frac{1}{m_j!} D_j^{s-1}\mathbf{F}^{(j)}(t_{m+})\varepsilon^s + \frac{1}{(m_j+1)!} D_j^{m_j}\mathbf{F}^{(j)}(t_{m+})\varepsilon^{m_j+1} + o(\varepsilon^{m_j+1}).\end{aligned}$$

In a similar fashion, one obtains

$$\begin{aligned}\mathbf{x}_{m-\varepsilon}^{(0)} &\equiv \mathbf{x}^{(0)}(t_m-\varepsilon)\\ &= \mathbf{x}^{(0)}(t_m) + \sum\nolimits_{s=1}^{2k_i} \frac{1}{s!} \frac{d^s \mathbf{x}^{(0)}}{dt^s}\bigg|_{t=t_m} (-\varepsilon)^s + \frac{1}{(2k_i+1)!} \frac{d^{2k_i+1} \mathbf{x}^{(0)}}{dt^{2k_{i+1}}}\bigg|_{t=t_m} (-\varepsilon)^{2k_i+1}\\ &\quad + o((-\varepsilon)^{2k_i+1})\\ &= \mathbf{x}^{(0)}(t_m) + \sum\nolimits_{s=1}^{2k_i} \frac{1}{s!} D_0^{s-1}\mathbf{F}^{(0)}(t_m)(-\varepsilon)^s + \frac{1}{(2k_i+1)!} D_0^{2k_i}\mathbf{F}^{(0)}(t_m)(-\varepsilon)^{2k_i+1}\\ &\quad + o((-\varepsilon)^{2k_i+1}),\end{aligned}$$

$$\begin{aligned}\mathbf{x}_{m+\varepsilon}^{(0)} &\equiv \mathbf{x}^{(0)}(t_m+\varepsilon)\\ &= \mathbf{x}^{(0)}(t_m) + \sum\nolimits_{s=1}^{m_j} \frac{1}{s!} \frac{d^s \mathbf{x}^{(0)}}{dt^s}\bigg|_{t=t_m} \varepsilon^s + \frac{1}{(m_j+1)!} \frac{d^{m_j+1} \mathbf{x}^{(0)}}{dt^{m_{j+1}}}\bigg|_{t=t_m} \varepsilon^{m_j+1} + o(\varepsilon^{m_j+1})\\ &= \mathbf{x}^{(0)}(t_m) + \sum\nolimits_{s=1}^{m_j} \frac{1}{s!} D_0^{s-1}\mathbf{F}^{(0)}(t_m)\varepsilon^s + \frac{1}{(m_j+1)!} D_0^{m_j}\mathbf{F}^{(0)}(t_m)\varepsilon^{m_j+1} + o(\varepsilon^{m_j+1}),\end{aligned}$$

and

$$\begin{aligned}\mathbf{n}_{\partial\Omega_{ij}}(\mathbf{x}_{m-\varepsilon}^{(0)}) &= \mathbf{n}_{\partial\Omega_{ij}}(\mathbf{x}_m^{(0)}) + \sum\nolimits_{s=1}^{2k_i}\frac{1}{s!}D_0^s\mathbf{n}_{\partial\Omega_{ij}}(\mathbf{x}_m^{(0)})(-\varepsilon)^s\\ &\quad+\frac{1}{(2k_i+1)!}D_0^{2k_i+1}\mathbf{n}_{\partial\Omega_{ij}}(\mathbf{x}_m^{(0)})(-\varepsilon)^{2k_i+1}+o((-\varepsilon)^{2k_i+1}),\\ \mathbf{n}_{\partial\Omega_{ij}}(\mathbf{x}_{m+\varepsilon}^{(0)}) &= \mathbf{n}_{\partial\Omega_{ij}}(\mathbf{x}_m^{(0)}) + \sum\nolimits_{s=1}^{m_j}\frac{1}{s!}D_0^s\mathbf{n}_{\partial\Omega_{ij}}(\mathbf{x}_m^{(0)})\varepsilon^s\\ &\quad+\frac{1}{(m_j+1)!}D_0^{m_j+1}\mathbf{n}_{\partial\Omega_{ij}}(\mathbf{x}_m^{(0)})\varepsilon^{m_j+1}+o\left(\varepsilon^{m_j+1}\right).\end{aligned}$$

The ignorance of the ε^{2k_i+2} and ε^{m_j+2}-terms and high order terms, the deformation of the above equation and left multiplication of $\mathbf{n}_{\partial\Omega_{ij}}$ gives

$$\begin{aligned}&\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}}(\mathbf{x}_{m-\varepsilon}^{(0)})\cdot[\mathbf{x}_{m-\varepsilon}^{(i)}-\mathbf{x}_{m-\varepsilon}^{(0)}]\\ &\quad=\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}}(\mathbf{x}_m^{(0)})\cdot[\mathbf{x}_{m-}^{(i)}-\mathbf{x}_m^{(0)}]+\sum\nolimits_{s=0}^{2k_i}\frac{1}{s!}(-\varepsilon)^s G_{\partial\Omega_{ij}}^{(s-1,i)}(\mathbf{x}_m,t_{m-},\mathbf{p}_i,\boldsymbol{\lambda})\\ &\quad+\frac{1}{(2k_i+1)!}(-\varepsilon)^{2k_i+1}G_{\partial\Omega_{ij}}^{(2k_i,i)}(\mathbf{x}_m,t_{m-},\mathbf{p}_i,\boldsymbol{\lambda}),\end{aligned}$$

$$\begin{aligned}&\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}}(\mathbf{x}_{m+\varepsilon}^{(0)})\cdot[\mathbf{x}_{m+\varepsilon}^{(j)}-\mathbf{x}_{m+\varepsilon}^{(0)}]\\ &\quad=\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}}(\mathbf{x}_m^{(0)})\cdot[\mathbf{x}_{m+}^{(j)}-\mathbf{x}_m^{(0)}]+\sum\nolimits_{s=1}^{m_j}\frac{1}{s!}\varepsilon^s G_{\partial\Omega_{ij}}^{(s-1,j)}(\mathbf{x}_m,t_{m+},\mathbf{p}_j,\boldsymbol{\lambda})\\ &\quad+\frac{1}{(m_j+1)!}\varepsilon^{m_j+1}G_{\partial\Omega_{ij}}^{(m_j,j)}(\mathbf{x}_m,t_{m+},\mathbf{p}_j,\boldsymbol{\lambda}).\end{aligned}$$

Because of $\mathbf{x}_{m\pm}^{(\alpha)}=\mathbf{x}_m^{(0)}=\mathbf{x}_m$, with Eqs. (3.35) and (3.36), one obtains

$$\begin{aligned}\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}}(\mathbf{x}_{m-\varepsilon}^{(0)})\cdot[\mathbf{x}_{m-\varepsilon}^{(0)}-\mathbf{x}_{m-\varepsilon}^{(i)}]&=\frac{1}{(2k_i+1)!}(-1)^{2k_i+2}\varepsilon^{2k_i+1}G_{\partial\Omega_{ij}}^{(2k_i,i)}(\mathbf{x}_m,t_{m-},\mathbf{p}_i,\boldsymbol{\lambda}),\\ \mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}}(\mathbf{x}_{m+\varepsilon}^{(0)})\cdot[\mathbf{x}_{m+\varepsilon}^{(j)}-\mathbf{x}_{m+\varepsilon}^{(0)}]&=\frac{1}{(m_j+1)!}\varepsilon^{m_j+1}G_{\partial\Omega_{ij}}^{(m_j,j)}(\mathbf{x}_m,t_{m+},\mathbf{p}_j,\boldsymbol{\lambda}).\end{aligned}$$

With Eq. (2.29), the foregoing equation gives Eq. (2.26). On the other hand, using Eq. (2.26), the foregoing equation gives Eq. (2.29). The proof is completed. □

2.4 Non-passable Flows

In this section, non-passable flows to a specific boundary will be discussed as in Luo [3, 7]. The initial discussion on such an issue can be found in Luo [5, 6]. The $(2k_i : 2k_j)$-non-passable flows are sketched in Fig. 2.5 for a better understanding of

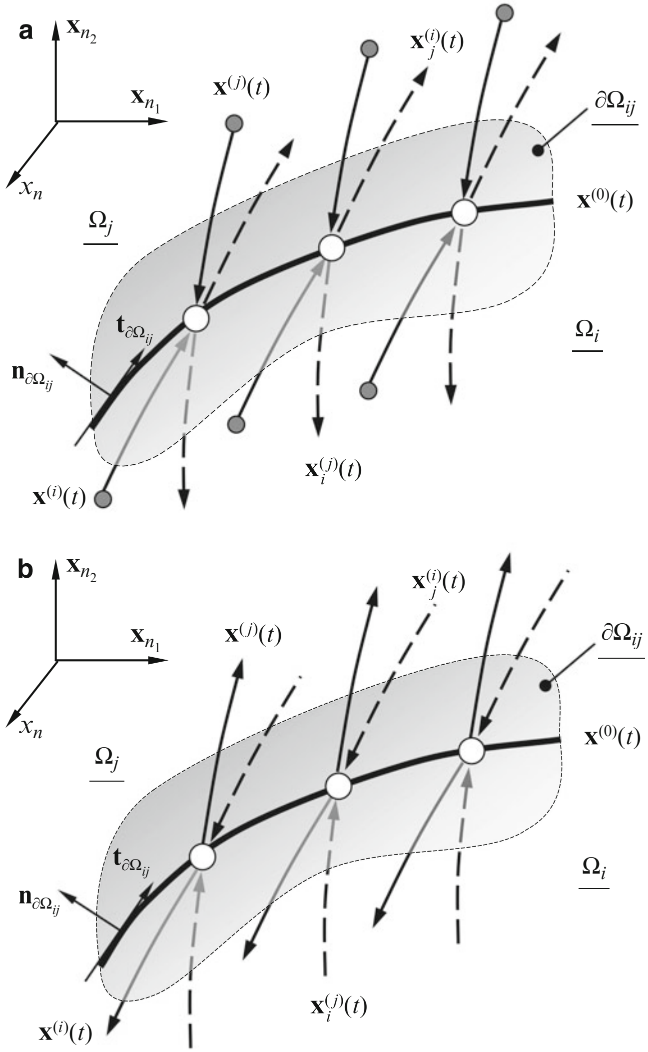

Fig. 2.5 The $(2k_i : 2k_j)$-non-passable flows: (**a**) the first kind (sink flows) and (**b**) the second kind (source flows). $\mathbf{x}^{(i)}(t)$ and $\mathbf{x}^{(j)}(t)$ represent *real* flows in domains Ω_i and Ω_j, respectively, which are depicted by *thin solid curves*. $\mathbf{x}_i^{(j)}(t)$ and $\mathbf{x}_j^{(i)}(t)$ represent *imaginary* flows in domains Ω_i and Ω_j, respectively, controlled by the vector fields in Ω_j and Ω_i, which are depicted by *dashed curves*. The flow on the boundary is described by $\mathbf{x}^{(0)}(t)$. The normal and tangential vectors $\mathbf{n}_{\partial\Omega_{ij}}$ and $\mathbf{t}_{\partial\Omega_{ij}}$ of the boundary are depicted. *Hollow circles* are for sink and source points on the boundary, and *filled circles* are for starting points $(n_1 + n_2 + 1 = n)$

non-passable flows. The non-passable flows are called the *full non-passable flows* because the flows on both sides of the boundary will approach or leave the boundary. If a flow only on one side of the boundary approaches or leaves the boundary, but the flow on the other side does not exist or is not defined, this flow to the boundary is called the *half-non-passable flow*. The full non-passable flow of the first kind (sink flows) and the full non-passable flow of the second kind (source flows) are sketched in Fig. 2.5a and b, respectively. The half-sink and half-source flow to the boundary will be discussed later.

Definition 2.13 For a discontinuous dynamic system in Eq. (2.1), there is a point $\mathbf{x}^{(0)}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ at time t_m between two adjacent domains Ω_α $(\alpha = i, j)$. For an arbitrarily small $\varepsilon > 0$, there is a time interval $[t_{m-\varepsilon}, t_m)$. Suppose $\mathbf{x}^{(\alpha)}(t_{m-}) = \mathbf{x}_m$ $(\alpha = i, j)$. The flow $\mathbf{x}^{(i)}(t)$ and $\mathbf{x}^{(j)}(t)$ to the boundary $\partial\Omega_{ij}$ is *non-passable* of the first kind (or called a sink flow) if

$$\left.\begin{array}{ll}\text{either} & \left.\begin{array}{l}\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}_{m-\varepsilon})\cdot[\mathbf{x}^{(0)}_{m-\varepsilon}-\mathbf{x}^{(i)}_{m-\varepsilon}]>0\\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}_{m-\varepsilon})\cdot[\mathbf{x}^{(0)}_{m-\varepsilon}-\mathbf{x}^{(j)}_{m-\varepsilon}]<0\end{array}\right\}\text{for } \mathbf{n}_{\partial\Omega_{ij}}\to\Omega_j\\ \text{or} & \left.\begin{array}{l}\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}_{m-\varepsilon})\cdot[\mathbf{x}^{(0)}_{m-\varepsilon}-\mathbf{x}^{(i)}_{m-\varepsilon}]<0\\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}_{m-\varepsilon})\cdot[\mathbf{x}^{(0)}_{m-\varepsilon}-\mathbf{x}^{(j)}_{m-\varepsilon}]>0\end{array}\right\}\text{for } \mathbf{n}_{\partial\Omega_{ij}}\to\Omega_i.\end{array}\right\} \tag{2.30}$$

From the foregoing definition, the sufficient and necessary conditions for the sink flow in Eq. (2.1) can be developed through the following theorem.

Theorem 2.3 *For a discontinuous dynamical system in Eq. (2.1), there is a point* $\mathbf{x}^{(0)}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ *at time* t_m *between two adjacent domains* Ω_α $(\alpha = i, j)$*. For an arbitrarily small* $\varepsilon > 0$*, there is a time interval* $[t_{m-\varepsilon}, t_m)$*. Suppose* $\mathbf{x}^{(\alpha)}(t_{m-}) = \mathbf{x}_m$ $(\alpha = i, j)$*. A flow* $\mathbf{x}^{(\alpha)}(t)$ *are* $C^{r_\alpha}_{[t_{m-\varepsilon}, t_m)}$*-continuous* $(r_\alpha \geq 1, \alpha = i, j)$ *for time t with* $||d^{r_\alpha+1}\mathbf{x}^{(\alpha)}/dt^{r_\alpha+1}|| < \infty$*. The flow* $\mathbf{x}^{(i)}(t)$ *and* $\mathbf{x}^{(j)}(t)$ *to the boundary* $\partial\Omega_{ij}$ *is non-passable of the first kind (or a sink flow) if and only if*

$$\begin{array}{ll}\text{either} & \left.\begin{array}{l}G^{(i)}_{\partial\Omega_{ij}}(\mathbf{x}_m, t_{m-}, \mathbf{p}_i, \boldsymbol{\lambda})>0\\ G^{(j)}_{\partial\Omega_{ij}}(\mathbf{x}_m, t_{m-}, \mathbf{p}_j, \boldsymbol{\lambda})<0\end{array}\right\}\text{ for } \mathbf{n}_{\partial\Omega_{ij}}\to\Omega_j,\\ \text{or} & \left.\begin{array}{l}G^{(i)}_{\partial\Omega_{ij}}(\mathbf{x}_m, t_{m-}, \mathbf{p}_i, \boldsymbol{\lambda})<0\\ G^{(j)}_{\partial\Omega_{ij}}(\mathbf{x}_m, t_{m-}, \mathbf{p}_j, \boldsymbol{\lambda})>0\end{array}\right\}\text{ for } \mathbf{n}_{\partial\Omega_{ij}}\to\Omega_i.\end{array} \tag{2.31}$$

Proof The proof is similar to the proof of Theorem 2.1. □

If the boundary $\partial\Omega_{ij}$ is independent of time, using Eq. (2.14), the above theorem is identical to results in Luo [5, 6] owing to the zero-order contact between the flow and boundary. However, in Luo [5, 6], a theory for the non-passable flow with the $(2k_i : 2k_j)$ higher order singularity $(k_\alpha \in \mathbf{N},\ \alpha = i, j)$ is only valid for the *plane* boundary and the $(2k_\alpha)$th-*contact* between the boundary $\partial\Omega_{ij}$ and the flow $\mathbf{x}^{(\alpha)}$ in the domain Ω_α $(\alpha = i, j)$, which will be discussed later. As in Luo [3, 7], with the higher order singularity of a flow to the boundary, a generalized theory for the $(2k_i : 2k_j)$-non-passable flow will be discussed herein.

Definition 2.14 For a discontinuous dynamic system in Eq. (2.1), there is a point $\mathbf{x}^{(0)}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ at time t_m between two adjacent domains Ω_α $(\alpha = i, j)$. For an arbitrarily small $\varepsilon > 0$, there is a time interval $[t_{m-\varepsilon}, t_m)$. Suppose $\mathbf{x}^{(\alpha)}(t_{m-}) = \mathbf{x}_m$ $(\alpha = i, j)$. A flow $\mathbf{x}^{(\alpha)}(t)$ is $C^{r_\alpha}_{[t_{m-\varepsilon}, t_m)}$-continuous $(r_\alpha \geq 2k_\alpha + 1,\ \alpha = i, j)$ for time t with $\|d^{r_\alpha+1}\mathbf{x}^{(\alpha)}/dt^{r_\alpha+1}\| < \infty$. The flow $\mathbf{x}^{(i)}(t)$ of the $(2k_i)$th-order and $\mathbf{x}^{(j)}(t)$ of the $(2k_j)$th-order to the boundary $\partial\Omega_{ij}$ is $(2k_i : 2k_j)$-*non-passable* of the first kind (or called a $(2k_i : 2k_j)$-sink flow) if

$$\left.\begin{aligned} &G^{(s_\alpha,\alpha)}_{\partial\Omega_{ij}}(\mathbf{x}_m, t_{m-}, \mathbf{p}_\alpha, \boldsymbol{\lambda}) = 0 \text{ for } s_\alpha = 0, 1, \cdots, 2k_\alpha - 1 \\ &G^{(2k_\alpha,\alpha)}_{\partial\Omega_{ij}}(\mathbf{x}_m, t_{m-}, \mathbf{p}_\alpha, \boldsymbol{\lambda}) \neq 0 \ (\alpha = i, j), \end{aligned}\right\} \tag{2.32}$$

$$\left.\begin{aligned} &\text{either} \left.\begin{aligned} &\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}_{m-\varepsilon}) \cdot [\mathbf{x}^{(0)}_{m-\varepsilon} - \mathbf{x}^{(i)}_{m-\varepsilon}] > 0 \\ &\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}_{m-\varepsilon}) \cdot [\mathbf{x}^{(0)}_{m-\varepsilon} - \mathbf{x}^{(j)}_{m-\varepsilon}] < 0 \end{aligned}\right\} \text{for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_j \\ &\text{or} \left.\begin{aligned} &\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}_{m-\varepsilon}) \cdot [\mathbf{x}^{(0)}_{m-\varepsilon} - \mathbf{x}^{(i)}_{m-\varepsilon}] < 0 \\ &\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}_{m-\varepsilon}) \cdot [\mathbf{x}^{(0)}_{m-\varepsilon} - \mathbf{x}^{(j)}_{m-\varepsilon}] > 0 \end{aligned}\right\} \text{for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_i. \end{aligned}\right\} \tag{2.33}$$

Theorem 2.4 *For a discontinuous dynamical system in Eq. (2.1), there is a point* $\mathbf{x}^{(0)}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ *at time* t_m *between two adjacent domains* Ω_α $(\alpha = i, j)$. *For an arbitrarily small* $\varepsilon > 0$, *there is a time interval* $[t_{m-\varepsilon}, t_m)$. *Suppose* $\mathbf{x}^{(\alpha)}(t_{m-}) = \mathbf{x}_m$ $(\alpha = i, j)$. *A flow* $\mathbf{x}^{(\alpha)}(t)$ *is* $C^{r_\alpha}_{[t_{m-\varepsilon}, t_m)}$*-continuous (*$r_\alpha \geq 2k_\alpha + 1$, $\alpha = i, j$*) for time* t *with* $\|d^{r_\alpha+1}\mathbf{x}^{(\alpha)}/dt^{r_\alpha+1}\| < \infty$. *The flow* $\mathbf{x}^{(i)}(t)$ *of the* $(2k_i)$th *order and* $\mathbf{x}^{(j)}(t)$ *of the* $(2k_j)$th *order to the boundary* $\partial\Omega_{ij}$ *is* $(2k_i : 2k_j)$*-non-passable of the first kind (or a* $(2k_i : 2k_j)$*-sink flow) if and only if*

$$G^{(s_\alpha,\alpha)}_{\partial\Omega_{ij}}(\mathbf{x}_m, t_{m-}, \mathbf{p}_\alpha, \boldsymbol{\lambda}) = 0 \quad \text{for } s_\alpha = 0, 1, \cdots, 2k_\alpha - 1 \text{ and } \alpha = i, j; \tag{2.34}$$

$$\begin{aligned} &\text{either} \left.\begin{aligned} &G^{(2k_i,i)}_{\partial\Omega_{ij}}(\mathbf{x}_m, t_{m-}, \mathbf{p}_i, \boldsymbol{\lambda}) > 0 \\ &G^{(2k_j,j)}_{\partial\Omega_{ij}}(\mathbf{x}_m, t_{m-}, \mathbf{p}_j, \boldsymbol{\lambda}) < 0 \end{aligned}\right\} \text{for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_j \\ &\text{or} \left.\begin{aligned} &G^{(2k_i,i)}_{\partial\Omega_{ij}}(\mathbf{x}_m, t_{m-}, \mathbf{p}_i, \boldsymbol{\lambda}) < 0 \\ &G^{(2k_j,j)}_{\partial\Omega_{ij}}(\mathbf{x}_m, t_{m-}, \mathbf{p}_j, \boldsymbol{\lambda}) > 0 \end{aligned}\right\} \text{for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_i. \end{aligned} \tag{2.35}$$

Proof The proof is similar to the proof of Theorem 2.2. □

Definition 2.15 For a discontinuous dynamic system in Eq. (2.1), there is a point $\mathbf{x}^{(0)}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ at time t_m between two adjacent domains Ω_α $(\alpha = i, j)$. For an arbitrarily small $\varepsilon > 0$, there is a time interval $(t_m, t_{m+\varepsilon}]$. Suppose $\mathbf{x}^{(\alpha)}(t_{m+}) = \mathbf{x}_m$ $(\alpha = i, j)$. The flow $\mathbf{x}^{(i)}(t)$ and $\mathbf{x}^{(j)}(t)$ to the boundary $\partial\Omega_{ij}$ is *non-passable* of the second kind (or called a source flow) if

$$\left.\begin{array}{ll}\text{either} & \left.\begin{array}{l}\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}_{m+\varepsilon}) \cdot [\mathbf{x}^{(i)}_{m+\varepsilon} - \mathbf{x}^{(0)}_{m+\varepsilon}] < 0 \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}_{m+\varepsilon}) \cdot [\mathbf{x}^{(j)}_{m+\varepsilon} - \mathbf{x}^{(0)}_{m+\varepsilon}] > 0\end{array}\right\} \text{for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_j \\ \text{or} & \left.\begin{array}{l}\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}_{m+\varepsilon}) \cdot [\mathbf{x}^{(i)}_{m+\varepsilon} - \mathbf{x}^{(0)}_{m+\varepsilon}] > 0 \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}_{m+\varepsilon}) \cdot [\mathbf{x}^{(j)}_{m+\varepsilon} - \mathbf{x}^{(0)}_{m+\varepsilon}] < 0\end{array}\right\} \text{for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_i.\end{array}\right\} \tag{2.36}$$

Theorem 2.5 *For a discontinuous dynamical system in Eq. (2.1), there is a point* $\mathbf{x}^{(0)}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ *at time* t_m *between two adjacent domains* Ω_α $(\alpha = i, j)$. *For an arbitrarily small* $\varepsilon > 0$, *there is a time interval* $(t_m, t_{m+\varepsilon}]$. *Suppose* $\mathbf{x}^{(\alpha)}(t_{m+}) = \mathbf{x}_m$ $(\alpha = i, j)$. *A flow* $\mathbf{x}^{(\alpha)}(t)$ *is* $C^{r_\alpha}_{(t_m, t_{m+\varepsilon}]}$*-continuous* $(r_\alpha \geq 1)$ *for time t with* $||d^{r_\alpha+1}\mathbf{x}^{(\alpha)}/dt^{r_\alpha+1}|| < \infty$. *The flow* $\mathbf{x}^{(i)}(t)$ *and* $\mathbf{x}^{(j)}(t)$ *to the boundary* $\partial\Omega_{ij}$ *is non-passable of the second kind (or a source flow) if and only if*

$$\left.\begin{array}{ll}\text{either} & \left.\begin{array}{l}G^{(i)}_{\partial\Omega_{ij}}(\mathbf{x}_m, t_{m+}, \mathbf{p}_i, \boldsymbol{\lambda}) < 0 \\ G^{(j)}_{\partial\Omega_{ij}}(\mathbf{x}_m, t_{m+}, \mathbf{p}_j, \boldsymbol{\lambda}) > 0\end{array}\right\} \text{ for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_j, \\ \text{or} & \left.\begin{array}{l}G^{(i)}_{\partial\Omega_{ij}}(\mathbf{x}_m, t_{m+}, \mathbf{p}_i, \boldsymbol{\lambda}) > 0 \\ G^{(j)}_{\partial\Omega_{ij}}(\mathbf{x}_m, t_{m+}, \mathbf{p}_j, \boldsymbol{\lambda}) < 0\end{array}\right\} \text{ for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_i.\end{array}\right\} \tag{2.37}$$

Proof The proof is similar to the proof of Theorem 2.1. □

Definition 2.16 For a discontinuous dynamic system in Eq. (2.1), there is a point $\mathbf{x}^{(0)}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ at time t_m between two adjacent domains Ω_α $(\alpha = i, j)$. For an arbitrarily small $\varepsilon > 0$, there is a time interval $(t_m, t_{m+\varepsilon}]$. Suppose $\mathbf{x}^{(\alpha)}(t_{m+}) = \mathbf{x}_m$ $(\alpha = i, j)$. A flow $\mathbf{x}^{(\alpha)}(t)$ is $C^{r_\alpha}_{(t_m, t_{m+\varepsilon}]}$-continuous $(r_\alpha \geq m_\alpha + 1, \alpha = i, j)$ for time t with $||d^{r_\alpha+1}\mathbf{x}^{(\alpha)}/dt^{r_\alpha+1}|| < \infty$. The flow $\mathbf{x}^{(i)}(t)$ of the m_ith-order and $\mathbf{x}^{(j)}(t)$ of the m_jth-order to the boundary $\partial\Omega_{ij}$ is $(m_i : m_j)$-*non-passable* of the second kind (or called an $(m_i : m_j)$-source flow) if

$$\left.\begin{array}{l}G^{(s_i, i)}_{\partial\Omega_{ij}}(\mathbf{x}_m, t_{m+}, \mathbf{p}_i, \boldsymbol{\lambda}) = 0 \text{ for } s_i = 0, 1, \cdots, m_i - 1 \\ G^{(2k_i, i)}_{\partial\Omega_{ij}}(\mathbf{x}_m, t_{m+}, \mathbf{p}_i, \boldsymbol{\lambda}) \neq 0,\end{array}\right\} \tag{2.38}$$

$$\left.\begin{aligned} &G^{(s_j,j)}_{\partial\Omega_{ij}}(\mathbf{x}_m, t_{m+}, \mathbf{p}_j, \boldsymbol{\lambda}) = 0 \text{ for } s_j = 0,1,\cdots,m_j-1 \\ &G^{(2k_j,j)}_{\partial\Omega_{ij}}(\mathbf{x}_m, t_{m+}, \mathbf{p}_j, \boldsymbol{\lambda}) \neq 0, \end{aligned}\right\} \tag{2.39}$$

$$\left.\begin{aligned} &\text{either} && \left.\begin{aligned} &\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}_{m+\varepsilon})\cdot[\mathbf{x}^{(i)}_{m+\varepsilon}-\mathbf{x}^{(0)}_{m+\varepsilon}]<0 \\ &\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}_{m+\varepsilon})\cdot[\mathbf{x}^{(j)}_{m+\varepsilon}-\mathbf{x}^{(0)}_{m+\varepsilon}]>0 \end{aligned}\right\}\text{for } \mathbf{n}_{\partial\Omega_{ij}}\to\Omega_j \\ &\text{or} && \left.\begin{aligned} &\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}_{m+\varepsilon})\cdot[\mathbf{x}^{(i)}_{m+\varepsilon}-\mathbf{x}^{(0)}_{m+\varepsilon}]>0 \\ &\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}_{m+\varepsilon})\cdot[\mathbf{x}^{(j)}_{m+\varepsilon}-\mathbf{x}^{(0)}_{m+\varepsilon}]<0 \end{aligned}\right\}\text{for } \mathbf{n}_{\partial\Omega_{ij}}\to\Omega_i. \end{aligned}\right\} \tag{2.40}$$

Note that for $m_\alpha = 2k_\alpha$ ($\alpha = i,j$), the $(2k_i : 2k_j)$-source flow is obtained, which corresponds to the $(2k_i : 2k_j)$-sink flow. If $m_\alpha = 2k_\alpha - 1$ and $m_\beta = 2k_\beta$ ($\alpha, \beta \in \{i,j\}$ and $\beta \neq \alpha$) or $m_\beta = 2k_\beta - 1$ ($\beta \in \{i,j\}$), because the source flow is from the boundary, three $(2k_i : 2k_j - 1), (2k_i - 1 : 2k_j)$, and $(2k_i - 1 : 2k_j - 1)$ source flows exist. However, the corresponding sink flows cannot be formed. Such source flows are relative to tangential flows, which will be discussed later. The question is which domain the flow will go into. If a source flow is exactly on the boundary, the source flow will keep on the boundary. However, if the flow just has a little bit perturbation on one of two domains (e.g., domain Ω_α, $\alpha \in \{i,j\}$), the source flow will get into the domain Ω_α with the corresponding flow of the m_αth-order. In fact, such a perturbed source flow is independent of the order of flow singularity to the boundary in another domain. One may say that the behavior of source flow is sensitive to the small perturbation in the vicinity of the boundary. Such a property is similar to the saddle or source points in continuous dynamical systems. However, the sink flow to the boundary is stabilized to the small perturbation to the boundary, which means the sink flow will be on the boundary whatever the perturbation of the sink flow is on the boundary or one of two domains. For a better explanation of the source flow, the four source flows are sketched in Figs. 2.6 and 2.7. Solid and dashed curves are for real and imaginary source flows. Dotted curves represent coming flows relative to the corresponding source flows to the boundary. In Fig. 2.6, $(2k_i : 2k_j)$ and $(2k_i - 1 : 2k_j - 1)$ source flows are presented. In fact, the source flow does not have any coming flow except for the source flow existing on the boundary. For a $(2k_i : 2k_j)$-source flow, the incoming flow is the imaginary flow, and for a $(2k_i - 1 : 2k_j - 1)$-source flow, the imagined, coming flow relative to the source flow is in the same domain. As in the $(2k_i : 2k_j)$-sink flow, the $(2k_i : 2k_j)$-source flow does not have any grazing properties to the boundary. However, the $(2k_i - 1 : 2k_j - 1)$-source flow possesses the grazing characteristics. Because the grazing source flows are not important for the flow passability to the boundary, the properties of grazing source flows will not be discussed in this section. In Fig. 2.7, the $(2k_i : 2k_j - 1)$ and $(2k_i - 1 : 2k_j)$-source flows are presented. From the foregoing definition, the sufficient and necessary conditions for the $(m_i : m_j)$-source flow in Eq. (2.1) can be developed as follows.

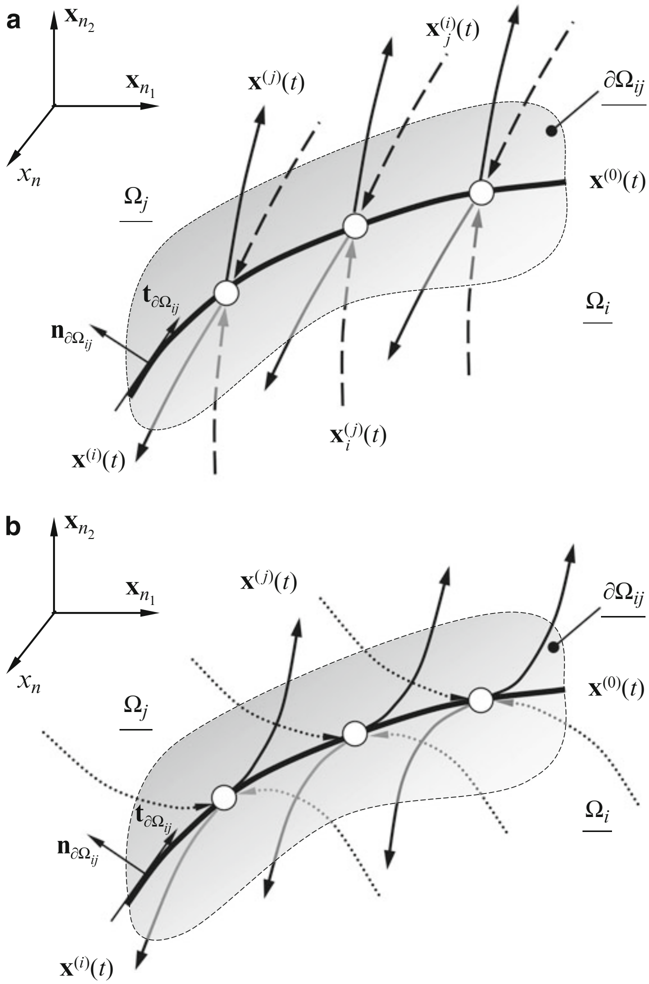

Fig. 2.6 Source flows: (**a**) the $(2k_i : 2k_j)$-source flows and (**b**) the $(2k_i - 1 : 2k_j - 1)$-source flows. $\mathbf{x}^{(i)}(t)$ and $\mathbf{x}^{(j)}(t)$ represent *real* flows in domains Ω_i and Ω_j, respectively, which are depicted by *thin solid curves*. $\mathbf{x}_i^{(j)}(t)$ and $\mathbf{x}_j^{(i)}(t)$ represent *imaginary* flows in domains Ω_i and Ω_j, respectively, controlled by the vector fields in Ω_j and Ω_i, which are depicted by *dashed curves*. The flow on the boundary is described by $\mathbf{x}^{(0)}(t)$. The normal and tangential vectors $\mathbf{n}_{\partial\Omega_{ij}}$ and $\mathbf{t}_{\partial\Omega_{ij}}$ of the boundary are depicted. *Hollow circles* are for source points on the boundary $(n_1 + n_2 + 1 = n)$

Theorem 2.6 *For a discontinuous dynamical system in Eq. (2.1), there is a point* $\mathbf{x}^{(0)}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ *at time* t_m *between two adjacent domains* Ω_α $(\alpha = i, j)$. *For an arbitrarily small* $\varepsilon > 0$, *there is a time interval* $(t_m, t_{m+\varepsilon}]$. *Suppose* $\mathbf{x}^{(\alpha)}(t_{m+}) = \mathbf{x}_m$ $(\alpha = i, j)$. *A flow* $\mathbf{x}^{(\alpha)}(t)$ *is* $C^{r_\alpha}_{(t_m, t_{m+\varepsilon}]}$*-continuous for time t with* $\|d^{r_\alpha+1}\mathbf{x}^{(\alpha)}/dt^{r_\alpha+1}\| < \infty\,(r_\alpha \geq m_\alpha + 1, \alpha = i, j)$. *The flow* $\mathbf{x}^{(i)}(t)$ *of the* m_ith-*order and*

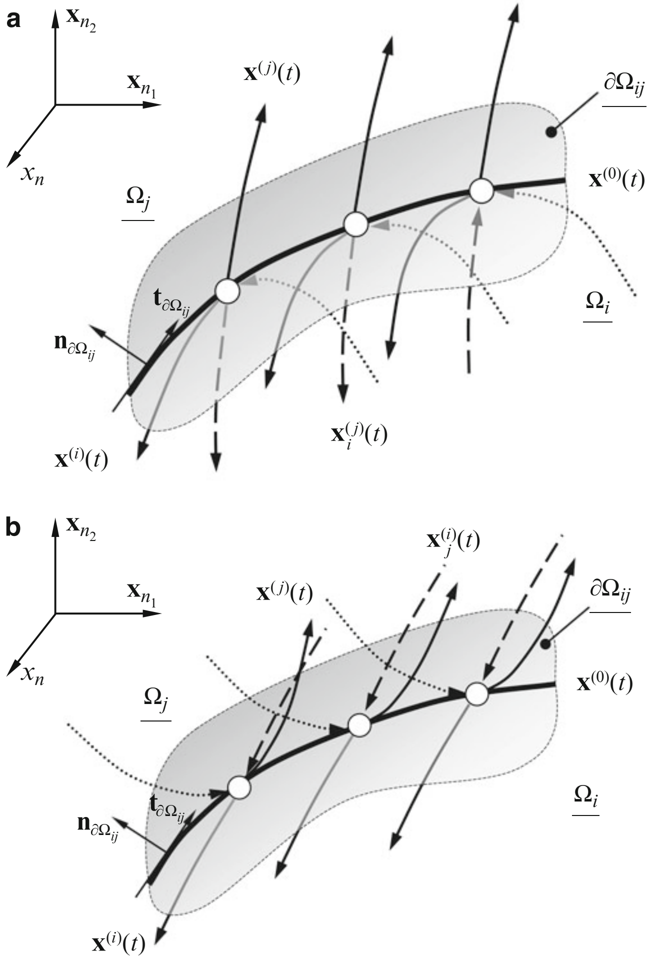

Fig. 2.7 Source flows: (**a**) the $(2k_i - 1 : 2k_j)$ source flows and (**b**) the $(2k_i : 2k_j - 1)$-source flows. $\mathbf{x}^{(i)}(t)$ and $\mathbf{x}^{(j)}(t)$ represent *real* flows in domains Ω_i and Ω_j, respectively, which are depicted by *thin solid curves*. $\mathbf{x}_i^{(j)}(t)$ and $\mathbf{x}_j^{(i)}(t)$ represent *imaginary* flows in domains Ω_i and Ω_j, respectively, controlled by the vector fields in Ω_j and Ω_i, which are depicted by *dashed curves*. The flow on the boundary is described by $\mathbf{x}^{(0)}(t)$. The normal and tangential vectors $\mathbf{n}_{\partial\Omega_{ij}}$ and $\mathbf{t}_{\partial\Omega_{ij}}$ of the boundary are depicted. *Hollow circles* are for source points on the boundary $(n_1 + n_2 + 1 = n)$

$\mathbf{x}^{(j)}(t)$ *of the* m_jth-*order to the boundary* $\partial\Omega_{ij}$ *is* $(m_i : m_j)$-*non-passable of the second kind* (*or* $(m_i : m_j)$-*source flow*) *if and only if*

$$G_{\partial\Omega_{ij}}^{(s_i,i)}(\mathbf{x}_m, t_{m+}, \mathbf{p}_i, \boldsymbol{\lambda}) = 0 \text{ for } s_i = 0, 1, \cdots, m_i - 1; \tag{2.41}$$

$$G^{(s_j,j)}_{\partial\Omega_{ij}}(\mathbf{x}_m, t_{m+}, \mathbf{p}_j, \boldsymbol{\lambda}) = 0 \text{ for } s_j = 0, 1, \cdots, m_j - 1; \tag{2.42}$$

$$\left.\begin{array}{ll} \text{either} & \left.\begin{array}{l} G^{(m_i,i)}_{\partial\Omega_{ij}}(\mathbf{x}_m, t_{m+}, \mathbf{p}_i, \boldsymbol{\lambda}) < 0 \\ G^{(m_j,j)}_{\partial\Omega_{ij}}(\mathbf{x}_m, t_{m+}, \mathbf{p}_j, \boldsymbol{\lambda}) > 0 \end{array}\right\} \text{ for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_j \\ \text{or} & \left.\begin{array}{l} G^{(m_i,i)}_{\partial\Omega_{ij}}(\mathbf{x}_m, t_{m+}, \mathbf{p}_i, \boldsymbol{\lambda}) > 0 \\ G^{(m_j,j)}_{\partial\Omega_{ij}}(\mathbf{x}_m, t_{m+}, \mathbf{p}_j, \boldsymbol{\lambda}) < 0 \end{array}\right\} \text{ for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_i. \end{array}\right. \tag{2.43}$$

Proof The proof is similar to the proof of Theorem 2.2. □

Next, half-non-passable flows to the boundary will be discussed. The half-non-passable flow of the first kind is termed *a half-sink flow*. A half-sink flow to the boundary is sketched in Fig. 2.8. Such a half-sink flow in $\boldsymbol{\Omega}_i$ is shown in Fig. 2.8a. Only $\mathbf{x}^{(i)}(t)$ for time $t \in [t_{m-\varepsilon}, t_m)$ is a real flow, and imaginary flows $\mathbf{x}^{(i)}_j(t)$ for time $t \in [t_{m-\varepsilon}, t_{m+\varepsilon}]$ and $\mathbf{x}^{(j)}_i(t)$ for time $t \in (t_m, t_{m+\varepsilon}]$ are represented by dashed curves. To the same boundary $\partial\boldsymbol{\Omega}_{ij}$, a half-sink flow in $\boldsymbol{\Omega}_j$ is sketched in Fig. 2.8b. The coming flow $\mathbf{x}^{(j)}(t)$ for time $t \in [t_{m-\varepsilon}, \varepsilon)$ is only a real flow. The strict mathematical description is given as follows.

Definition 2.17 For a discontinuous dynamical system in Eq. (2.1), there is a point $\mathbf{x}^{(0)}(t_m) \equiv \mathbf{x}_m \in \partial\boldsymbol{\Omega}_{ij}$ at time t_m between two adjacent domains $\boldsymbol{\Omega}_\alpha(\alpha = i, j)$. Suppose $\mathbf{x}^{(i)}(t_{m-}) = \mathbf{x}_m = \mathbf{x}^{(j)}_i(t_{m\pm})$. For an arbitrarily small $\varepsilon > 0$, there are two time intervals $[t_{m-\varepsilon}, t_m)$ and $[t_{m-\varepsilon}, t_{m+\varepsilon}]$. A flow $\mathbf{x}^{(i)}(t)$ is $C^{r_i}_{[t_{m-\varepsilon}, t_m)}$-continuous $(r_i \geq 2k_i + 1)$ with $\|d^{r_i+1}\mathbf{x}^{(i)}/dt^{r_i+1}\| < \infty$ for time t, and an imaginary flow $\mathbf{x}^{(j)}_i(t)$ is $C^{r_j}_{[t_{m-\varepsilon}, t_{m+\varepsilon}]}$-continuous $(r_j \geq 2k_j)$ and $\|d^{r_j+1}\mathbf{x}^{(j)}_i/dt^{r_j+1}\| < \infty$. The flow $\mathbf{x}^{(i)}(t)$ of the $(2k_i)$th-order and $\mathbf{x}^{(j)}_i(t)$ of the $(2k_j - 1)$th-order to the boundary $\partial\boldsymbol{\Omega}_{ij}$ is $(2k_i : 2k_j - 1)$-*half-non-passable* of the first kind in domain $\boldsymbol{\Omega}_i$ (or called a $(2k_i : 2k_j - 1)$-half-sink flow) if

$$\left.\begin{array}{l} G^{(s_i,i)}_{\partial\Omega_{ij}}(\mathbf{x}_m, t_{m-}, \mathbf{p}_i, \boldsymbol{\lambda}) = 0 \text{ for } s_i = 0, 1, \cdots, 2k_i - 1 \\ G^{(2k_i,i)}_{\partial\Omega_{ij}}(\mathbf{x}_m, t_{m-}, \mathbf{p}_i, \boldsymbol{\lambda}) \neq 0, \end{array}\right\} \tag{2.44}$$

$$\left.\begin{array}{l} G^{(s_j,j)}_{\partial\Omega_{ij}}(\mathbf{x}_m, t_{m\pm}, \mathbf{p}_j, \boldsymbol{\lambda}) = 0 \text{ for } s_j = 0, 1, \cdots, 2k_j - 2 \\ G^{(2k_j-1,j)}_{\partial\Omega_{ij}}(\mathbf{x}_m, t_{m\pm}, \mathbf{p}_j, \boldsymbol{\lambda}) \neq 0, \end{array}\right\} \tag{2.45}$$

$$\left.\begin{array}{ll} \text{either} & \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}_{m-\varepsilon}) \cdot [\mathbf{x}^{(0)}_{m-\varepsilon} - \mathbf{x}^{(i)}_{m-\varepsilon}] > 0 \text{ for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_j \\ \text{or} & \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}_{m-\varepsilon}) \cdot [\mathbf{x}^{(0)}_{m-\varepsilon} - \mathbf{x}^{(i)}_{m-\varepsilon}] < 0 \text{ for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_i, \end{array}\right\} \tag{2.46}$$

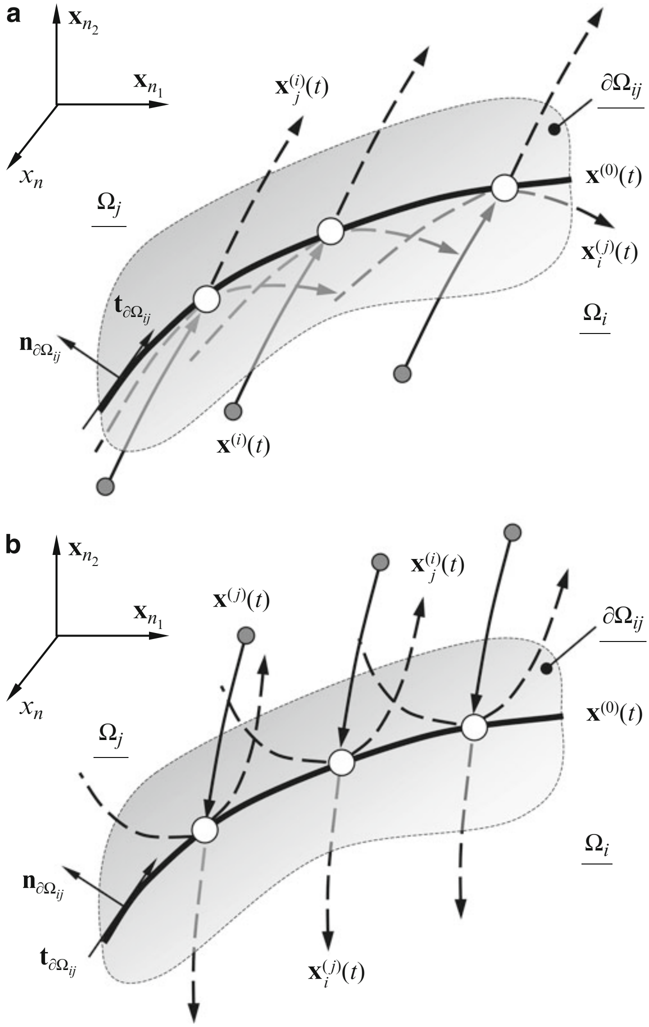

Fig. 2.8 The half-sink flows: (**a**) $(2k_i : 2k_j - 1)$-order in Ω_i and (**b**) $(2k_j : 2k_i - 1)$-order in Ω_j. $\mathbf{x}^{(i)}(t)$ and $\mathbf{x}^{(j)}(t)$ represent *real* flows in domains Ω_i and Ω_j, respectively, which are depicted by *thin solid curves*. $\mathbf{x}_i^{(j)}(t)$ and $\mathbf{x}_j^{(i)}(t)$ represent *imaginary* flows in domains Ω_i and Ω_j, respectively, controlled by the vector fields in Ω_j and Ω_i, which are depicted by *dashed curves*. The flow on the boundary is described by $\mathbf{x}^{(0)}(t)$. The normal and tangential vectors $\mathbf{n}_{\partial\Omega_{ij}}$ and $\mathbf{t}_{\partial\Omega_{ij}}$ on the boundary are depicted. *Hollow circles* are for sink points on the boundary and *filled circles* are for starting points $(n_1 + n_2 + 1 = n)$

$$\left.\begin{array}{ll}\text{either} & \left.\begin{array}{l}\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}_{m-\varepsilon})\cdot[\mathbf{x}^{(0)}_{m-\varepsilon}-\mathbf{x}^{(j)}_{i(m-\varepsilon)}]>0\\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}_{m+\varepsilon})\cdot[\mathbf{x}^{(j)}_{i(m+\varepsilon)}-\mathbf{x}^{(0)}_{m+\varepsilon}]<0\end{array}\right\}\text{ for } \mathbf{n}_{\partial\Omega_{ij}}\to\Omega_j\\ \text{or} & \left.\begin{array}{l}\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}_{m-\varepsilon})\cdot[\mathbf{x}^{(0)}_{m-\varepsilon}-\mathbf{x}^{(j)}_{i(m-\varepsilon)}]<0\\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}_{m+\varepsilon})\cdot[\mathbf{x}^{(j)}_{i(m+\varepsilon)}-\mathbf{x}^{(0)}_{m+\varepsilon}]>0\end{array}\right\}\text{ for } \mathbf{n}_{\partial\Omega_{ij}}\to\Omega_i.\end{array}\right\} \tag{2.47}$$

Theorem 2.7 *For a discontinuous dynamical system in Eq. (2.1), there is a point* $\mathbf{x}^{(0)}(t_m)\equiv\mathbf{x}_m\in\partial\Omega_{ij}$ *at time* t_m *between two adjacent domains* Ω_α $(\alpha=i,j)$. *Suppose* $\mathbf{x}^{(i)}(t_{m-})=\mathbf{x}_m=\mathbf{x}_i^{(j)}(t_{m\pm})$. *For an arbitrarily small* $\varepsilon>0$, *there are two time intervals* $[t_{m-\varepsilon},t_m)$ *and* $[t_{m-\varepsilon},t_{m+\varepsilon}]$. *A flow* $\mathbf{x}^{(i)}(t)$ *is* $C^{r_i}_{[t_{m-\varepsilon},t_m)}$*-continuous* $(r_i\geq 2k_i+1)$ *with* $||d^{r_i+1}\mathbf{x}^{(i)}/dt^{r_i+1}||<\infty$ *for time t, and an imaginary flow* $\mathbf{x}_i^{(j)}(t)$ *is* $C^{r_j}_{[t_{m-\varepsilon},t_{m+\varepsilon}]}$*-continuous* $(r_j\geq 2k_j)$ *with* $||d^{r_j+1}\mathbf{x}_i^{(j)}/dt^{r_j+1}||<\infty$. *The flow* $\mathbf{x}^{(i)}(t)$ *of the* $(2k_i)$th-*order and* $\mathbf{x}_i^{(j)}(t)$ *of the* $(2k_j-1)$th-*order to the boundary* $\partial\Omega_{ij}$ *is* $(2k_i:2k_j-1)$-*half-non-passable of the first kind in domain* Ω_i *(or a* $(2k_i:2k_j-1)$-*half-sink flow) if and only if*

$$G^{(s_i,i)}_{\partial\Omega_{ij}}(\mathbf{x}_m,t_{m-},\mathbf{p}_i,\boldsymbol{\lambda})=0\quad\text{for } s_i=0,1,\cdots,2k_i-1; \tag{2.48}$$

$$G^{(s_j,j)}_{\partial\Omega_{ij}}(\mathbf{x}_m,t_{m\pm},\mathbf{p}_j,\boldsymbol{\lambda})=0\quad\text{for } s_j=0,1,\cdots,2k_j-2; \tag{2.49}$$

$$\left.\begin{array}{ll}\text{either} & \left.\begin{array}{l}G^{(2k_i,i)}_{\partial\Omega_{ij}}(\mathbf{x}_m,t_{m-},\mathbf{p}_i,\boldsymbol{\lambda})>0\\ G^{(2k_j-1,j)}_{\partial\Omega_{ij}}(\mathbf{x}_m,t_{m\pm},\mathbf{p}_j,\boldsymbol{\lambda})<0\end{array}\right\}\text{ for } \mathbf{n}_{\partial\Omega_{ij}}\to\Omega_j\\ \text{or} & \left.\begin{array}{l}G^{(2k_i,i)}_{\partial\Omega_{ij}}(\mathbf{x}_m,t_{m-},\mathbf{p}_i,\boldsymbol{\lambda})<0\\ G^{(2k_j-1,j)}_{\partial\Omega_{ij}}(\mathbf{x}_m,t_{m\pm},\mathbf{p}_j,\boldsymbol{\lambda})>0\end{array}\right\}\text{ for } \mathbf{n}_{\partial\Omega_{ij}}\to\Omega_i.\end{array}\right\} \tag{2.50}$$

Proof The proof is similar to the proof of Theorem 2.2. □

Before the half-non-passable flow of the second kind is discussed, the intuitive illustration of the half-non-passable flow is sketched in Figs. 2.9 and 2.10 for a better understanding of this concept. The half-non-passable flow of the second kind is termed *a half-source flow*. The half-source flows in Ω_i are presented in Fig. 2.9a. $\mathbf{x}^{(i)}(t)$ for time $t\in(t_m,t_{m+\varepsilon}]$ is only a real flow. The imaginary flows $\mathbf{x}_j^{(i)}(t)$ for time $t\in[t_{m-\varepsilon},t_{m+\varepsilon}]$ and $\mathbf{x}_i^{(j)}(t)$ for time $t\in[t_{m-\varepsilon},t_m)$ are represented by dashed curves. To the same boundary $\partial\Omega_{ij}$, a half-source flow in Ω_j is sketched in Fig. 2.9b. The leaving flow $\mathbf{x}^{(j)}(t)$ for $t\in(t_m,t_{m+\varepsilon}]$ is a real flow. Similarly, the $(2k_i-1:2k_j-1)$-half-source flow in domain Ω_i and Ω_j will be presented in Fig. 2.10a and b, respectively.

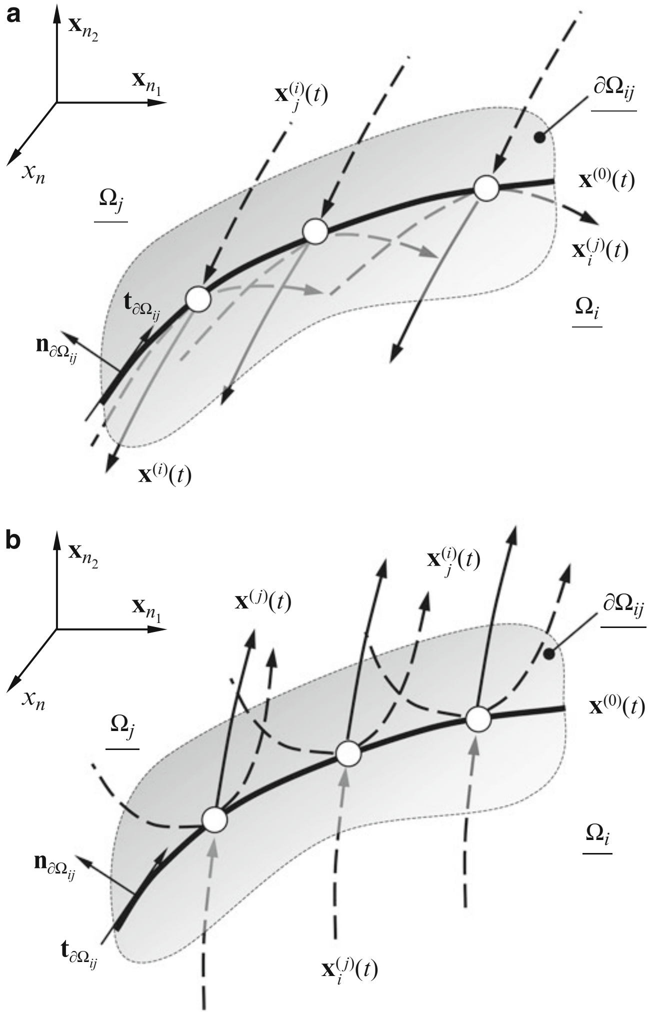

Fig. 2.9 Half-source flows: (**a**) $(2k_i : 2k_j - 1)$-order in Ω_i and (**b**) $(2k_i - 1 : 2k_j)$-order in Ω_j. $\mathbf{x}^{(i)}(t)$ and $\mathbf{x}^{(j)}(t)$ represent *real* flows in domains Ω_i and Ω_j, respectively, which are depicted by *thin solid curves*. $\mathbf{x}_i^{(j)}(t)$ and $\mathbf{x}_j^{(i)}(t)$ represent *imaginary* flows in domains Ω_i and Ω_j, respectively, controlled by the vector fields in Ω_j and Ω_i, which are depicted by *dashed curves*. The flow on the boundary is described by $\mathbf{x}^{(0)}(t)$. The normal and tangential vectors $\mathbf{n}_{\partial\Omega_{ij}}$ and $\mathbf{t}_{\partial\Omega_{ij}}$ of the boundary are depicted. *Hollow circles* are for source points on the boundary $(n_1 + n_2 + 1 = n)$

Definition 2.18 For a discontinuous dynamical system in Eq. (2.1), there is a point $\mathbf{x}^{(0)}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ at time t_m between two adjacent domains $\Omega_\alpha(\alpha = i, j)$. For an arbitrarily small $\varepsilon > 0$, there are two time intervals $[t_{m-\varepsilon}, t_m)$ and $[t_{m-\varepsilon}, t_{m+\varepsilon}]$. Suppose $\mathbf{x}^{(\alpha)}(t_{m+}) = \mathbf{x}_m = \mathbf{x}_\alpha^{(\beta)}(t_{m\pm})$. A flow $\mathbf{x}^{(\alpha)}(t)$ is $C^{r_\alpha}_{[t_{m-\varepsilon}, t_m)}$-continuous for time t with $\|d^{r_\alpha+1}\mathbf{x}^{(\alpha)}/dt^{r_\alpha+1}\| < \infty$ $(r_\alpha \geq m_\alpha + 1)$, and an imaginary flow $\mathbf{x}_\alpha^{(\beta)}(t)$ is

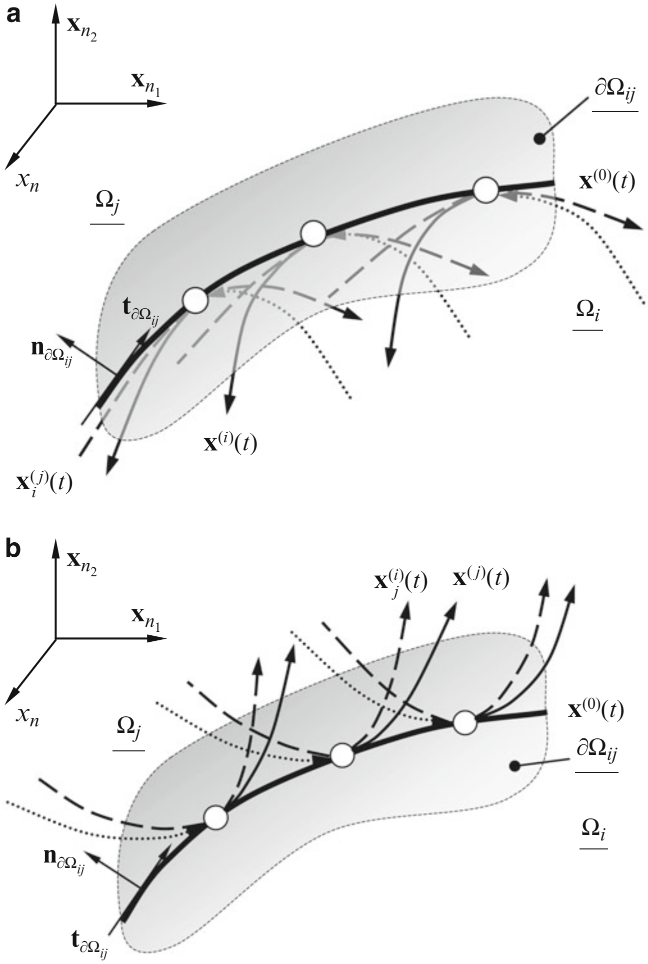

Fig. 2.10 $(2k_i - 1 : 2k_j - 1)$-half-source flows in: (**a**) domain Ω_i and (**b**) domain Ω_j. $\mathbf{x}^{(i)}(t)$ and $\mathbf{x}^{(j)}(t)$ represent *real* flows in domains Ω_i and Ω_j, respectively, which are depicted by *thin solid curves*. $\mathbf{x}_i^{(j)}(t)$ and $\mathbf{x}_j^{(i)}(t)$ represent *imaginary* flows in domains Ω_i and Ω_j, respectively, controlled by the vector fields in Ω_j and Ω_i, which are depicted by *dashed curves*. The flow on the boundary is described by $\mathbf{x}^{(0)}(t)$. The normal and tangential vectors $\mathbf{n}_{\partial\Omega_{ij}}$ and $\mathbf{t}_{\partial\Omega_{ij}}$ of the boundary are depicted. *Hollow circles* are for source points on the boundary $(n_1 + n_2 + 1 = n)$

$C^{r_\beta}_{[t_{m-\varepsilon}, t_{m+\varepsilon}]}$-continuous with $\|d^{r_\beta+1}\mathbf{x}_\alpha^{(\beta)}/dt^{r_\beta+1}\| < \infty$ ($r_\beta \geq 2k_\beta$, $\beta = i, j$ and $\beta \neq \alpha$). The flow $\mathbf{x}^{(\alpha)}(t)$ of the m_αth-order and $\mathbf{x}_\alpha^{(\beta)}(t)$ of the $(2k_\beta - 1)$th-order to the boundary $\partial\Omega_{ij}$ is $(m_\alpha : 2k_\beta - 1)$-half-*non-passable* of the second kind in domain Ω_α (or called an $(m_\alpha : 2k_\beta - 1)$-half-source flow) if

$$\left.\begin{array}{l} G^{(s_\alpha,\alpha)}_{\partial\Omega_{ij}}(\mathbf{x}_m, t_{m+}, \mathbf{p}_\alpha, \boldsymbol{\lambda}) = 0 \text{ for } s_\alpha = 0, 1, \cdots, m_\alpha - 1 \\ G^{(2k_\alpha,\alpha)}_{\partial\Omega_{ij}}(\mathbf{x}_m, t_{m+}, \mathbf{p}_\alpha, \boldsymbol{\lambda}) \neq 0, \end{array}\right\} \tag{2.51}$$

$$\left.\begin{aligned} &G^{(s_\beta,\beta)}_{\partial\Omega_{ij}}(\mathbf{x}_m, t_{m\pm}, \mathbf{p}_\beta, \boldsymbol{\lambda}) = 0 \text{ for } s_\beta = 0,1,\cdots,2k_\beta - 2 \\ &G^{(2k_\beta-1,\beta)}_{\partial\Omega_{ij}}(\mathbf{x}_m, t_{m\pm}, \mathbf{p}_\beta, \boldsymbol{\lambda}) \neq 0, \end{aligned}\right\} \tag{2.52}$$

$$\left.\begin{aligned} &\text{either } \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}_{m+\varepsilon}) \cdot [\mathbf{x}^{(\alpha)}_{m+\varepsilon} - \mathbf{x}^{(0)}_{m+\varepsilon}] < 0 \text{ for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_\beta \\ &\text{or } \quad \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}_{m+\varepsilon}) \cdot [\mathbf{x}^{(\alpha)}_{m+\varepsilon} - \mathbf{x}^{(0)}_{m+\varepsilon}] > 0 \text{ for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_\alpha, \end{aligned}\right\} \tag{2.53}$$

$$\left.\begin{aligned} &\text{either} \left.\begin{aligned} &\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}_{m-\varepsilon}) \cdot [\mathbf{x}^{(0)}_{m-\varepsilon} - \mathbf{x}^{(\beta)}_{\alpha(m-\varepsilon)}] > 0 \\ &\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}_{m+\varepsilon}) \cdot [\mathbf{x}^{(\beta)}_{\alpha(m+\varepsilon)} - \mathbf{x}^{(0)}_{m+\varepsilon}] < 0 \end{aligned}\right\} \text{for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_\beta \\ &\text{or} \left.\begin{aligned} &\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}_{m-\varepsilon}) \cdot [\mathbf{x}^{(0)}_{m-\varepsilon} - \mathbf{x}^{(\beta)}_{\alpha(m-\varepsilon)}] < 0 \\ &\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}_{m+\varepsilon}) \cdot [\mathbf{x}^{(\beta)}_{\alpha(m+\varepsilon)} - \mathbf{x}^{(0)}_{m+\varepsilon}] > 0 \end{aligned}\right\} \text{for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_\alpha. \end{aligned}\right\} \tag{2.54}$$

From the above definition, the necessary and sufficient conditions for such a $(m_\alpha : 2k_\beta - 1)$ half-non-passable flow of the second kind (or $(m_\alpha : 2k_\beta - 1)$ half-source flow) are stated in the following theorem.

Theorem 2.8 *For a discontinuous dynamical system in Eq. (2.1), there is a point $\mathbf{x}^{(0)}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ at time t_m between two adjacent domains Ω_α ($\alpha = i, j$). For an arbitrarily small $\varepsilon > 0$, there are two time intervals $[t_{m-\varepsilon}, t_m)$ and $[t_{m-\varepsilon}, t_{m+\varepsilon}]$. Suppose $\mathbf{x}^{(\alpha)}(t_{m+}) = \mathbf{x}_m = \mathbf{x}^{(\beta)}_\alpha(t_{m\pm})$. A flow $\mathbf{x}^{(\alpha)}(t)$ is $C^{r_\alpha}_{[t_{m-\varepsilon}, t_m)}$-continuous for time t with $\|d^{r_\alpha+1}\mathbf{x}^{(\alpha)}/dt^{r_\alpha+1}\| < \infty$ ($r_\alpha \geq m_\alpha + 1$), and an imaginary flow $\mathbf{x}^{(\beta)}_\alpha(t)$ is $C^{r_\beta}_{[t_{m-\varepsilon}, t_{m+\varepsilon}]}$-continuous with $\|d^{r_\beta+1}\mathbf{x}^{(\beta)}_\alpha/dt^{r_\beta+1}\| < \infty$ ($r_\beta \geq 2k_\beta$). The flow $\mathbf{x}^{(\alpha)}(t)$ of the* m_αth-*order and $\mathbf{x}^{(\beta)}_\alpha(t)$ of the $(2k_\beta - 1)$*th-*order to the boundary $\partial\Omega_{ij}$ is $(m_\alpha: 2k_\beta - 1)$-half-non-passable of the second kind in domain Ω_α (or an $(m_\alpha : 2k_\beta - 1)$-half-source flow) if and only if*

$$G^{(s_\alpha,\alpha)}_{\partial\Omega_{ij}}(\mathbf{x}_m, t_{m+}, \mathbf{p}_\alpha, \boldsymbol{\lambda}) = 0 \text{ for } s_\alpha = 0,1,\cdots,m_\alpha - 1; \tag{2.55}$$

$$G^{(s_\beta,\beta)}_{\partial\Omega_{ij}}(\mathbf{x}_m, t_{m\pm}, \mathbf{p}_\beta, \boldsymbol{\lambda}) = 0 \text{ for } s_\beta = 0,1,\cdots,2k_\beta - 2 \tag{2.56}$$

$$\left.\begin{aligned} &\text{either} \left.\begin{aligned} &G^{(2k_\alpha,\alpha)}_{\partial\Omega_{ij}}(\mathbf{x}_m, t_{m+}, \mathbf{p}_\alpha, \boldsymbol{\lambda}) < 0 \\ &G^{(2k_\beta-1,\beta)}_{\partial\Omega_{ij}}(\mathbf{x}_m, t_{m\pm}, \mathbf{p}_\beta, \boldsymbol{\lambda}) < 0 \end{aligned}\right\} \text{for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_\beta \\ &\text{or} \left.\begin{aligned} &G^{(2k_\alpha,\alpha)}_{\partial\Omega_{ij}}(\mathbf{x}_m, t_{m+}, \mathbf{p}_\alpha, \boldsymbol{\lambda}) > 0 \\ &G^{(2k_\alpha,\alpha)}_{\partial\Omega_{ij}}(\mathbf{x}_m, t_{m+}, \mathbf{p}_\alpha, \boldsymbol{\lambda}) > 0 \end{aligned}\right\} \text{for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_\alpha. \end{aligned}\right\} \tag{2.57}$$

Proof The proof is similar to the proof of Theorem 2.2. □

2.5 Grazing Flows

The tangency of a flow to the boundary in a discontinuous dynamical system is called the grazing, which also includes the imaginary flows tangential to the boundary.

Definition 2.19 For a discontinuous dynamical system in Eq. (2.1), there is a point $\mathbf{x}^{(0)}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ at time t_m between two adjacent domains Ω_α $(\alpha = i, j)$. Suppose $\mathbf{x}^{(\alpha)}(t_{m\pm}) = \mathbf{x}_m$ $(\alpha \in \{i, j\})$. For an arbitrarily small $\varepsilon > 0$, there is a time interval $[t_{m-\varepsilon}, t_{m+\varepsilon}]$. A flow $\mathbf{x}^{(\alpha)}(t)$ is $C^{r_\alpha}_{[t_{m-\varepsilon}, t_{m+\varepsilon}]}$-continuous $(r_\alpha \geq 2)$ for time t. The flow $\mathbf{x}^{(\alpha)}(t)$ in domain Ω_α is *tangential* to the boundary $\partial\Omega_{ij}$ if

$$G^{(\alpha)}_{\partial\Omega_{ij}}(\mathbf{x}_m, t_m, \mathbf{p}_\alpha, \boldsymbol{\lambda}) = 0 \text{ and } G^{(1,\alpha)}_{\partial\Omega_{ij}}(\mathbf{x}_m, t_m, \mathbf{p}_\alpha, \boldsymbol{\lambda}) \neq 0; \tag{2.58}$$

$$\begin{aligned} &\text{either} \quad \left.\begin{aligned} \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}_{m-\varepsilon}) \cdot [\mathbf{x}^{(0)}_{m-\varepsilon} - \mathbf{x}^{(\alpha)}_{m-\varepsilon}] > 0 \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}_{m+\varepsilon}) \cdot [\mathbf{x}^{(\alpha)}_{m+\varepsilon} - \mathbf{x}^{(0)}_{m+\varepsilon}] < 0 \end{aligned}\right\} \text{ for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_\beta \\ &\text{or} \quad \left.\begin{aligned} \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}_{m-\varepsilon}) \cdot [\mathbf{x}^{(0)}_{m-\varepsilon} - \mathbf{x}^{(\alpha)}_{m-\varepsilon}] < 0 \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}_{m+\varepsilon}) \cdot [\mathbf{x}^{(\alpha)}_{m+\varepsilon} - \mathbf{x}^{(0)}_{m+\varepsilon}] > 0 \end{aligned}\right\} \text{ for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_\alpha. \end{aligned} \tag{2.59}$$

Theorem 2.9 *For a discontinuous dynamical system in Eq. (2.1), there is a point* $\mathbf{x}^{(0)}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ *at time* t_m *between two adjacent domains* Ω_α $(\alpha = i, j)$. *Suppose* $\mathbf{x}^{(\alpha)}(t_{m\pm}) = \mathbf{x}_m$ $(\alpha \in \{i, j\})$. *For an arbitrarily small* $\varepsilon > 0$, *there is a time interval* $[t_{m-\varepsilon}, t_{m+\varepsilon}]$. *A flow* $\mathbf{x}^{(\alpha)}(t)$ *is* $C^{r}_{[t_{m-\varepsilon}, t_{m+\varepsilon}]}$*-continuous* $(r \geq 2)$ *for time t with* $||d^{r_\alpha+1}\mathbf{x}^{(\alpha)}/dt^{r_\alpha+1}|| < \infty$. *The flow* $\mathbf{x}^{(\alpha)}(t)$ *in domain* Ω_α *is tangential to the boundary* $\partial\Omega_{ij}$ *if and only if*

$$G^{(0,\alpha)}_{\partial\Omega_{ij}}(\mathbf{x}_m, t_m, \mathbf{p}_\alpha, \boldsymbol{\lambda}) = 0 \text{ for } \alpha \in \{i, j\}; \tag{2.60}$$

$$\begin{aligned} &\text{either } G^{(1,\alpha)}_{\partial\Omega_{ij}}(\mathbf{x}_m, t_m, \mathbf{p}_\alpha, \boldsymbol{\lambda}) < 0 \text{ for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_\beta \\ &\text{or} \qquad G^{(1,\alpha)}_{\partial\Omega_{ij}}(\mathbf{x}_m, t_m, \mathbf{p}_\alpha, \boldsymbol{\lambda}) > 0 \text{ for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_\alpha. \end{aligned} \tag{2.61}$$

Proof Equation (2.60) is identical to Eq. (2.58), thus the condition in Eq. (2.58) is satisfied, and vice versa. Suppose $\mathbf{x}^{(\alpha)}(t_{m\pm}) = \mathbf{x}_m$ $(\alpha \in \{i, j\})$ and $\mathbf{x}^{(\alpha)}(t)$ are $C^{r_\alpha}_{[t_{m-\varepsilon}, t_{m+\varepsilon}]}$-continuous $(r_\alpha \geq 2)$ for time t and $||d^{r_\alpha}\mathbf{x}^{(\alpha)}/dt^{r_\alpha}|| < \infty$ $(\alpha \in \{i, j\})$. For $a \in [t_{m-\varepsilon}, t_m)$ or $a \in (t_m, t_{m+\varepsilon}]$, the Taylor series expansion of $\mathbf{x}^{(\alpha)}(t_{m\pm\varepsilon})$ to $\mathbf{x}^{(\alpha)}(a)$ up to the third-order term gives

$$\begin{aligned}\mathbf{x}^{(\alpha)}_{m\pm\varepsilon} &\equiv \mathbf{x}^{(\alpha)}(t_{m\pm} \pm \varepsilon)\\ &= \mathbf{x}^{(\alpha)}|_{t=a} + \dot{\mathbf{x}}^{(\alpha)}|_{t=a}(t_{m\pm} \pm \varepsilon - a) + \frac{1}{2!}\ddot{\mathbf{x}}^{(\alpha)}|_{t=a}(t_{m\pm} \pm \varepsilon - a)^2\\ &\quad + o((t_{m\pm} \pm \varepsilon - a)^2).\end{aligned}$$

As $a \to t_{m\pm}$, taking the limit of the foregoing equation leads to

$$\mathbf{x}^{(\alpha)}_{m\pm\varepsilon} \equiv \mathbf{x}^{(\alpha)}(t_m \pm \varepsilon) = \mathbf{x}^{(\alpha)}_{m\pm} \pm \dot{\mathbf{x}}^{(\alpha)}_{m\pm}\varepsilon + \frac{1}{2!}\ddot{\mathbf{x}}^{(\alpha)}_{m\pm}\varepsilon^2 + o(\varepsilon^2).$$

In a similar fashion, we have

$$\begin{aligned}\mathbf{x}^{(0)}_{m\pm\varepsilon} &\equiv \mathbf{x}^{(0)}(t_m \pm \varepsilon) = \mathbf{x}^{(0)}_{m} \pm \dot{\mathbf{x}}^{(0)}_{m}\varepsilon + \frac{1}{2!}\ddot{\mathbf{x}}^{(0)}_{m}\varepsilon^2 + o(\varepsilon^2),\\ \mathbf{n}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}_{m\pm\varepsilon}) &\equiv \mathbf{n}_{\partial\Omega_{ij}}|_{\mathbf{x}^{(0)}_m} \pm D_0\mathbf{n}_{\partial\Omega_{ij}}|_{\mathbf{x}^{(0)}_m}\varepsilon + \frac{1}{2!}D_0^2\mathbf{n}_{\partial\Omega_{ij}}|_{\mathbf{x}^{(0)}_m}\varepsilon^2 + o(\varepsilon^2).\end{aligned}$$

The ignorance of the ε^3 and high order terms, the deformation of the above equation and left multiplication of $\mathbf{n}_{\partial\Omega_{ij}}$ gives

$$\begin{aligned}\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}_{m\pm\varepsilon})\cdot[\mathbf{x}^{(\alpha)}_{m\pm\varepsilon} - \mathbf{x}^{(0)}_{m\pm\varepsilon}] = \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}_{m})\cdot[\mathbf{x}^{(\alpha)}_{m\pm} - \mathbf{x}^{(0)}_{m}]\\ \pm \varepsilon G^{(0,\alpha)}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}_m, \mathbf{x}^{(\alpha)}_{m\pm}, t_m, \mathbf{p}_\alpha, \boldsymbol{\lambda})\\ + \frac{1}{2!}\varepsilon^2 G^{(1,\alpha)}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}_m, \mathbf{x}^{(\alpha)}_{m\pm}, t_m, \mathbf{p}_\alpha, \boldsymbol{\lambda}).\end{aligned}$$

Due to $\mathbf{x}^{(\alpha)}_{m\pm} = \mathbf{x}^{(0)}_m = \mathbf{x}_m$ and $G^{(0,\alpha)}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}_m, \mathbf{x}^{(\alpha)}_{m\pm}, t_m, \mathbf{p}_\alpha, \boldsymbol{\lambda}) \equiv G^{(0,\alpha)}_{\partial\Omega_{ij}}(\mathbf{x}_m, t_m, \mathbf{p}_\alpha, \boldsymbol{\lambda}) = 0$, the foregoing equation becomes

$$\begin{aligned}\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}_{m-\varepsilon})\cdot[\mathbf{x}^{(0)}_{m-\varepsilon} - \mathbf{x}^{(\alpha)}_{m-\varepsilon}] &= -\frac{1}{2!}\varepsilon^2 G^{(1,\alpha)}_{\partial\Omega_{ij}}(\mathbf{x}_m, t_m, \mathbf{p}_\alpha, \boldsymbol{\lambda}),\\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}_{m+\varepsilon})\cdot[\mathbf{x}^{(\alpha)}_{m+\varepsilon} - \mathbf{x}^{(0)}_{m+\varepsilon}] &= \frac{1}{2!}\varepsilon^2 G^{(1,\alpha)}_{\partial\Omega_{ij}}(\mathbf{x}_m, t_m, \mathbf{p}_\alpha, \boldsymbol{\lambda}).\end{aligned}$$

Using Eq. (2.61), Eq. (2.59) is obtained. On the other hand, using Eq. (2.59), Eq. (2.61) is achieved. Therefore, this theorem is proved. □

Definition 2.20 For a discontinuous dynamical system in Eq. (2.1), there is a point $\mathbf{x}^{(0)}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ at time t_m between two adjacent domains Ω_α ($\alpha = i, j$). Suppose $\mathbf{x}^{(\alpha)}(t_{m\pm}) = \mathbf{x}_m$ ($\alpha \in \{i, j\}$). For an arbitrarily small $\varepsilon > 0$, there is a time interval $[t_{m-\varepsilon}, t_{m+\varepsilon}]$. A flow $\mathbf{x}^{(\alpha)}(t)$ is $C^{r_\alpha}_{[t_{m-\varepsilon}, t_{m+\varepsilon}]}$-continuous ($r_\alpha \geq k_\alpha + 1$) with $||d^{r_\alpha+1}\mathbf{x}^{(\alpha)}/dt^{r_\alpha+1}|| < \infty$ for time t. A flow $\mathbf{x}^{(\alpha)}(t)$ in Ω_α is *tangential* to the boundary $\partial\Omega_{ij}$ of the $(2k_\alpha - 1)$th-order if

$$\begin{aligned}&G^{(s_\alpha,\alpha)}_{\partial\Omega_{ij}}(\mathbf{x}_m, t_m, \mathbf{p}_\alpha, \boldsymbol{\lambda}) = 0 \text{ for } s_\alpha = 0, 1, \cdots, 2k_\alpha - 2,\\ &G^{(2k_\alpha-1,\alpha)}_{\partial\Omega_{ij}}(\mathbf{x}_m, t_m, \mathbf{p}_\alpha, \boldsymbol{\lambda}) \neq 0;\end{aligned} \tag{2.62}$$

$$\left.\begin{array}{ll}\text{either} & \left.\begin{array}{l}\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}_{m-\varepsilon})\cdot[\mathbf{x}^{(0)}_{m-\varepsilon}-\mathbf{x}^{(\alpha)}_{m-\varepsilon}]>0\\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}_{m+\varepsilon})\cdot[\mathbf{x}^{(\alpha)}_{m+\varepsilon}-\mathbf{x}^{(0)}_{m+\varepsilon}]<0\end{array}\right\}\text{for } \mathbf{n}_{\partial\Omega_{ij}}\to\Omega_\beta\\ \text{or} & \left.\begin{array}{l}\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}_{m-\varepsilon})\cdot[\mathbf{x}^{(0)}_{m-\varepsilon}-\mathbf{x}^{(\alpha)}_{m-\varepsilon}]<0\\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}_{m+\varepsilon})\cdot[\mathbf{x}^{(\alpha)}_{m+\varepsilon}-\mathbf{x}^{(0)}_{m+\varepsilon}]>0\end{array}\right\}\text{for } \mathbf{n}_{\partial\Omega_{ij}}\to\Omega_\alpha.\end{array}\right. \tag{2.63}$$

Theorem 2.10 *For a discontinuous dynamical system in Eq. (2.1), there is a point* $\mathbf{x}^{(0)}(t_m)\equiv\mathbf{x}_m\in\partial\Omega_{ij}$ *at time* t_m *between two adjacent domains* Ω_α *(*$\alpha=i,j$*). Suppose* $\mathbf{x}^{(\alpha)}(t_{m\pm})=\mathbf{x}_m$ *(*$\alpha\in\{i,j\}$*). For an arbitrarily small* $\varepsilon>0$*, there is a time interval* $[t_{m-\varepsilon},t_{m+\varepsilon}]$*. A flow* $\mathbf{x}^{(\alpha)}(t)$ *is* $C^{r_\alpha}_{[t_{m-\varepsilon},t_{m+\varepsilon}]}$*-continuous (*$r_\alpha\geq k_\alpha+1$*) for time t with* $||d^{r_\alpha+1}\mathbf{x}^{(\alpha)}/dt^{r_\alpha+1}||<\infty$*. A flow* $\mathbf{x}^{(\alpha)}(t)$ *in* Ω_α *is tangential to the boundary* $\partial\Omega_{ij}$ *of the* $(2k_\alpha-1)$th-*order if and only if*

$$G^{(s_\alpha,\alpha)}_{\partial\Omega_{ij}}(\mathbf{x}_m,t_m,\mathbf{p}_\alpha,\boldsymbol{\lambda})=0 \text{ for } s_\alpha=0,1,\cdots,2k_\alpha-2, \tag{2.64}$$

$$\begin{array}{ll}\text{either} & G^{(2k_\alpha-1,\alpha)}_{\partial\Omega_{ij}}(\mathbf{x}_m,t_m,\mathbf{p}_\alpha,\boldsymbol{\lambda})<0 \text{ for } \mathbf{n}_{\partial\Omega_{ij}}\to\Omega_\beta\\ \text{or} & G^{(2k_\alpha-1,\alpha)}_{\partial\Omega_{ij}}(\mathbf{x}_m,t_m,\mathbf{p}_\alpha,\boldsymbol{\lambda})>0 \text{ for } \mathbf{n}_{\partial\Omega_{ij}}\to\Omega_\alpha.\end{array} \tag{2.65}$$

Proof Equation (2.64) is identical to Eq. (2.62), thus the condition in Eq. (2.64) is satisfied, and vice versa. Suppose $\mathbf{x}^{(\alpha)}(t_{m\pm})=\mathbf{x}_m$ $(\alpha\in\{i,j\})$ and $\mathbf{x}^{(\alpha)}(t)$ are $C^{r_\alpha}_{[t_{m-\varepsilon},t_{m+\varepsilon}]}$-continuous ($r_\alpha\geq 2k_\alpha+1$) for time t and $||d^{r_\alpha}\mathbf{x}^{(\alpha)}/dt^{r_\alpha}||<\infty$ ($\alpha\in\{i,j\}$). For $a\in[t_{m-\varepsilon},t_m)$ or $a\in(t_m,t_{m+\varepsilon}]$, the Taylor series expansion of $\mathbf{x}^{(\alpha)}(t_{m\pm\varepsilon})$ to $\mathbf{x}^{(\alpha)}(a)$ up to the $(2k_\alpha+1)$th-order term gives

$$\begin{aligned}\mathbf{x}^{(\alpha)}_{m\pm\varepsilon}&\equiv\mathbf{x}^{(\alpha)}(t_{m\pm}\pm\varepsilon)\\&=\mathbf{x}^{(\alpha)}(a)+\sum\nolimits_{s_\alpha=1}^{2k_\alpha-1}\frac{1}{s_\alpha!}\frac{d^{s_\alpha}\mathbf{x}^{(\alpha)}}{dt^{s_\alpha}}\bigg|_{t=a}.(t_{m\pm}\pm\varepsilon-a)^{s_\alpha}+\frac{1}{(2k_\alpha)!}\frac{d^{2k_\alpha}\mathbf{x}^{(\alpha)}}{dt^{2k_\alpha}}\bigg|_{t=a}\\&\quad\times(t_{m\pm}\pm\varepsilon-a)^{2k_\alpha}+o((t_{m\pm}\pm\varepsilon-a)^{2k_\alpha}).\end{aligned}$$

As $a\to t_{m\pm}$, taking the limit of the foregoing equation leads to

$$\begin{aligned}\mathbf{x}^{(\alpha)}_{m\pm\varepsilon}&\equiv\mathbf{x}^{(\alpha)}(t_m\pm\varepsilon)\\&=\mathbf{x}^{(\alpha)}_{m\pm}+\sum\nolimits_{s_\alpha=1}^{2k_\alpha-1}\frac{1}{s_\alpha!}\frac{d^{s_\alpha}\mathbf{x}^{(\alpha)}}{dt^{s_\alpha}}\bigg|_{\mathbf{x}^{(\alpha)}_{m\pm}}.(\pm\varepsilon)^{s_\alpha}+\frac{1}{(2k_\alpha)!}\frac{d^{2k_\alpha}\mathbf{x}^{(\alpha)}}{dt^{2k_\alpha}}\bigg|_{\mathbf{x}^{(\alpha)}_{m\pm}}.(\pm\varepsilon)^{2k_\alpha}+o(\varepsilon^{2k_\alpha}).\end{aligned}$$

In a similar fashion, one obtains

$$\begin{aligned}\mathbf{x}_{m\pm\varepsilon}^{(0)} &\equiv \mathbf{x}^{(0)}(t_m \pm \varepsilon)\\ &= \mathbf{x}_m^{(0)} + \sum\nolimits_{s_\alpha=1}^{2k_\alpha-1} \frac{1}{s_\alpha!} \frac{d^{s_\alpha}\mathbf{x}^{(0)}}{dt^{s_\alpha}}\bigg|_{\mathbf{x}_m^{(0)}} \cdot (\pm\varepsilon)^{s_\alpha} + \frac{1}{(2k_\alpha)!} \frac{d^{2k_\alpha}\mathbf{x}^{(0)}}{dt^{2k_\alpha}}\bigg|_{\mathbf{x}_m^{(0)}} \cdot \varepsilon^{2k_\alpha} + o(\varepsilon^{2k_\alpha}),\end{aligned}$$

$$\begin{aligned}\mathbf{n}_{\partial\Omega_{ij}}(\mathbf{x}_{m\pm\varepsilon}^{(0)}) &\equiv \mathbf{n}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}(t_{m\pm\varepsilon}))\\ &= \mathbf{n}_{\partial\Omega_{ij}}(\mathbf{x}_m^{(0)}) + \sum\nolimits_{s_\alpha=1}^{2k_\alpha-1} \frac{1}{s_\alpha!} D_{\mathbf{x}^{(0)}}^{s_\alpha} \mathbf{n}_{\partial\Omega_{ij}}\Big|_{\mathbf{x}_m^{(0)}} (\pm\varepsilon)^{s_\alpha}\\ &\quad + \frac{1}{(2k_\alpha)!} D_{\mathbf{x}^{(0)}}^{2k_\alpha} \mathbf{n}_{\partial\Omega_{ij}}\Big|_{\mathbf{x}_m^{(0)}} \varepsilon^{2k_\alpha} + o(\varepsilon^{2k_\alpha}).\end{aligned}$$

The ignorance of the $\varepsilon^{2k_\alpha+1}$ and high order terms, the deformation of the above equation and left multiplication of $\mathbf{n}_{\partial\Omega_{ij}}$ gives

$$\begin{aligned}\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}}(\mathbf{x}_{m\pm\varepsilon}^{(0)}) \cdot [\mathbf{x}_{m\pm\varepsilon}^{(\alpha)} - \mathbf{x}_{m\pm\varepsilon}^{(0)}] &= \mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}}(\mathbf{x}_m^{(0)}) \cdot [\mathbf{x}_{m\pm}^{(\alpha)} - \mathbf{x}_m^{(0)}]\\ &\quad + \sum\nolimits_{s_\alpha=1}^{2k_\alpha-1} \frac{1}{s_\alpha!} (\pm\varepsilon)^{s_\alpha} G_{\partial\Omega_{ij}}^{(s_\alpha-1,\alpha)}(\mathbf{x}_m^{(0)}, \mathbf{x}_{m\pm}^{(\alpha)}, t_m, \mathbf{p}_\alpha, \boldsymbol{\lambda})\\ &\quad + \frac{1}{(2k_\alpha)!} \varepsilon^{2k_\alpha} G_{\partial\Omega_{ij}}^{(2k_\alpha-1,\alpha)}(\mathbf{x}_m^{(0)}, \mathbf{x}_{m\pm}^{(\alpha)}, t_m, \mathbf{p}_\alpha, \boldsymbol{\lambda}).\end{aligned}$$

Due to $\mathbf{x}_{m\pm}^{(\alpha)} = \mathbf{x}_m^{(0)} = \mathbf{x}_m$ and $G_{\partial\Omega_{ij}}^{(s_\alpha,\alpha)}(\mathbf{x}_m^{(0)}, \mathbf{x}_m^{(\alpha)}, t_m, \mathbf{p}_\alpha, \boldsymbol{\lambda}) \equiv G_{\partial\Omega_{ij}}^{(s_\alpha,\alpha)}(\mathbf{x}_m, t_m, \mathbf{p}_\alpha, \boldsymbol{\lambda}) = 0$ for $s_\alpha = 0, 1, \cdots, 2k_\alpha - 2$, the foregoing equation becomes

$$\begin{aligned}\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}}(\mathbf{x}_{m-\varepsilon}^{(0)}) \cdot [\mathbf{x}_{m-\varepsilon}^{(0)} - \mathbf{x}_{m-\varepsilon}^{(\alpha)}] &= -\frac{1}{(2k_\alpha)!} \varepsilon^{2k_\alpha} G_{\partial\Omega_{ij}}^{(2k_\alpha-1,\alpha)}(\mathbf{x}_m, t_m, \mathbf{p}_\alpha, \boldsymbol{\lambda}),\\ \mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}}(\mathbf{x}_{m+\varepsilon}^{(0)}) \cdot [\mathbf{x}_{m+\varepsilon}^{(\alpha)} - \mathbf{x}_{m+\varepsilon}^{(0)}] &= \frac{1}{(2k_\alpha)!} \varepsilon^{2k_\alpha} G_{\partial\Omega_{ij}}^{(2k_\alpha-1,\alpha)}(\mathbf{x}_m, t_m, \mathbf{p}_\alpha, \boldsymbol{\lambda}).\end{aligned}$$

Using Eq. (2.63), Eq. (2.65) is obtained. However, using Eq. (2.65), equation (2.63) is obtained. Therefore, this theorem is proved. □

The flow grazing bifurcation to the boundary can be determined by the G-function $G_{\partial\Omega_{ij}}^{(2k_\alpha-1,\alpha)}(\mathbf{x}_m, t_m, \mathbf{p}_\alpha, \boldsymbol{\lambda})$. In other words, the conditions for a flow tangential to the boundary are $G_{\partial\Omega_{ij}}^{(s_\alpha,\alpha)}(\mathbf{x}_m, t_m, \mathbf{p}_\alpha, \boldsymbol{\lambda}) = 0$ ($s_\alpha = 0, 1, \cdots, 2k_\alpha - 2$) and $G_{\partial\Omega_{ij}}^{(2k_\alpha-1,\alpha)}(\mathbf{x}_m, t_m, \mathbf{p}_\alpha, \boldsymbol{\lambda}) < 0$ (or $G_{\partial\Omega_{ij}}^{(2k_\alpha-1,\alpha)}(\mathbf{x}_m, t_m, \mathbf{p}_\alpha, \boldsymbol{\lambda}) > 0$) for the boundary $\partial\Omega_{ij}$ with $\mathbf{n}_{\partial\Omega_{ij}} \to \Omega_j$ (or $\mathbf{n}_{\partial\Omega_{ij}} \to \Omega_i$). To develop a uniform theory of the tangential flow with the passable and non-passable flow, the imaginary flow tangency will be introduced. To distinguish a real tangential flow from an imaginary tangential flow, the tangency of a real flow to the boundary can be restated as follows.

Definition 2.21 For a discontinuous dynamical system in Eq. (2.1), there is a point $\mathbf{x}^{(0)}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ at time t_m between two adjacent domains Ω_α ($\alpha = i, j$). Suppose $\mathbf{x}^{(i)}(t_{m+}) = \mathbf{x}_m = \mathbf{x}_i^{(j)}(t_{m\pm})$. For an arbitrarily small $\varepsilon > 0$, there are two time intervals $[t_{m-\varepsilon}, t_m)$ and $[t_{m-\varepsilon}, t_{m+\varepsilon}]$. A flow $\mathbf{x}^{(i)}(t)$ is $C^{r_i}_{[t_{m-\varepsilon},t_{m+\varepsilon}]}$-continuous ($r_i \geq 2k_i + 1$) for time t and $\|d^{r_i+1}\mathbf{x}^{(i)}/dt^{r_i+1}\| < \infty$, and a flow $\mathbf{x}^{(j)}(t)$ is $C^{r_j}_{[t_{m-\varepsilon},t_m)}$ or $C^{r_j}_{(t_m,t_{m+\varepsilon}]}$-continuous with $\|d^{r_j+1}\mathbf{x}^{(j)}/dt^{r_j+1}\| < \infty$ ($r_j \geq 2k_j$). The flow $\mathbf{x}^{(i)}(t)$ of the $(2k_i-1)$th-order with $\mathbf{x}^{(j)}(t)$ of the $(2k_j)$th-order to the boundary $\partial\Omega_{ij}$ is a $(2k_i-1:2k_j)$-tangential flow in domain Ω_i if

$$\left.\begin{array}{l} G^{(s_i,i)}_{\partial\Omega_{ij}}(\mathbf{x}_m, t_{m\pm}, \mathbf{p}_i, \boldsymbol{\lambda}) = 0 \text{ for } s_i = 0, 1, \cdots, 2k_i - 2 \\ G^{(2k_i,i)}_{\partial\Omega_{ij}}(\mathbf{x}_m, t_{m\pm}, \mathbf{p}_i, \boldsymbol{\lambda}) \neq 0, \end{array}\right\} \tag{2.66}$$

$$\left.\begin{array}{l} G^{(s_j,j)}_{\partial\Omega_{ij}}(\mathbf{x}_m, t_{m\pm}, \mathbf{p}_j, \boldsymbol{\lambda}) = 0 \text{ for } s_j = 0, 1, \cdots, 2k_j - 1 \\ G^{(2k_j-1,j)}_{\partial\Omega_{ij}}(\mathbf{x}_m, t_{m\pm}, \mathbf{p}_j, \boldsymbol{\lambda}) \neq 0, \end{array}\right\} \tag{2.67}$$

$$\begin{array}{ll} \text{either} & \left.\begin{array}{l} \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}_{m-\varepsilon}) \cdot [\mathbf{x}^{(0)}_{m-\varepsilon} - \mathbf{x}^{(i)}_{m-\varepsilon}] > 0 \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}_{m+\varepsilon}) \cdot [\mathbf{x}^{(i)}_{m+\varepsilon} - \mathbf{x}^{(0)}_{m+\varepsilon}] < 0 \end{array}\right\} \text{for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_j \\ \text{or} & \left.\begin{array}{l} \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}_{m-\varepsilon}) \cdot [\mathbf{x}^{(0)}_{m-\varepsilon} - \mathbf{x}^{(i)}_{m-\varepsilon}] < 0 \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}_{m+\varepsilon}) \cdot [\mathbf{x}^{(i)}_{m+\varepsilon} - \mathbf{x}^{(0)}_{m+\varepsilon}] > 0 \end{array}\right\} \text{for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_i, \end{array} \tag{2.68}$$

$$\begin{array}{ll} \text{either} & \left.\begin{array}{l} \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}_{m-\varepsilon}) \cdot [\mathbf{x}^{(0)}_{m-\varepsilon} - \mathbf{x}^{(j)}_{m-\varepsilon}] < 0 \text{ or} \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}_{m+\varepsilon}) \cdot [\mathbf{x}^{(j)}_{m+\varepsilon} - \mathbf{x}^{(0)}_{m+\varepsilon}] > 0 \end{array}\right\} \text{for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_j \\ \text{or} & \left.\begin{array}{l} \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}_{m-\varepsilon}) \cdot [\mathbf{x}^{(0)}_{m-\varepsilon} - \mathbf{x}^{(j)}_{m-\varepsilon}] > 0 \text{ or} \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}_{m+\varepsilon}) \cdot [\mathbf{x}^{(j)}_{m+\varepsilon} - \mathbf{x}^{(0)}_{m+\varepsilon}] < 0 \end{array}\right\} \text{for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_i. \end{array} \tag{2.69}$$

To explain the foregoing definition, such a $(2k_i-1:2k_j)$-tangential flow to the boundary $\partial\Omega_{ij}$ in domain Ω_i is sketched in Fig. 2.11a with source in domain Ω_j and (b) with sink in domain Ω_j. The $(2k_j-1:2k_i)$-tangential flow in domain Ω_j is sketched in Fig. 2.12a with source in domain Ω_i and (b) with sink in domain Ω_i. The sink and source flows are represented by the dotted curves. The tangential flows are presented by solid curves. The dashed curves denote the imaginary flows. If the starting point is on the flow $\mathbf{x}^{(j)}(t)$ (or $\mathbf{x}^{(j)}_i(t)$) in Fig. 2.11b (or Fig. 2.12a), the passable flow from domain Ω_j to Ω_i (or Ω_j to Ω_i) is formed. Such passable flows possess the post-higher-order singularity. From the above definition, the necessary and sufficient conditions for the tangential flow are given in the following theorem.

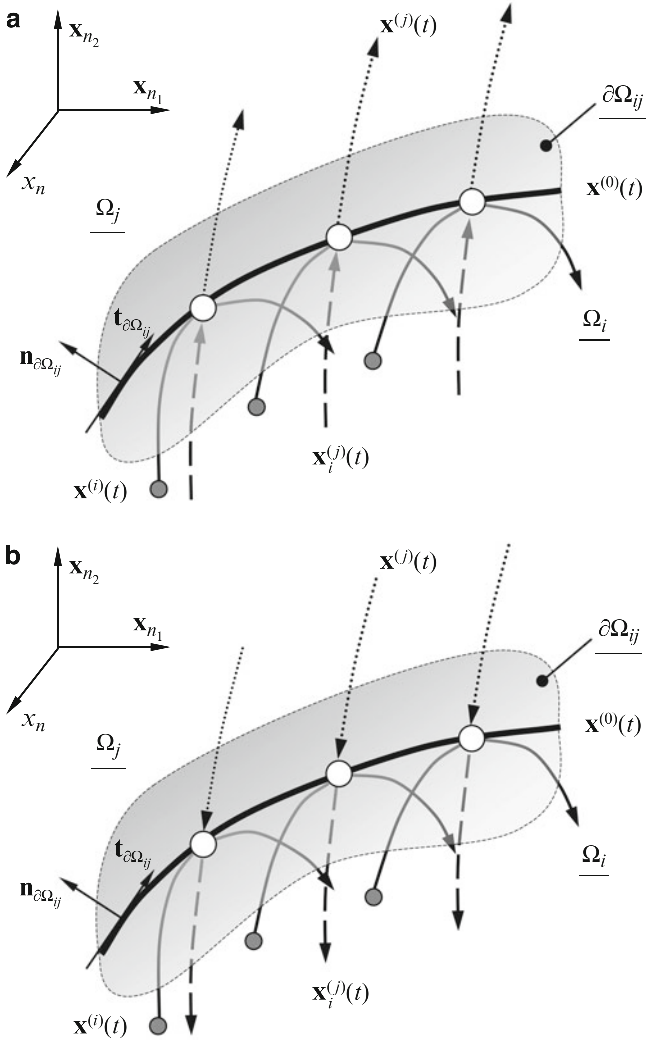

Fig. 2.11 The $(2k_i - 1 : 2k_j)$-tangential flows in Ω_i: (**a**) with source in Ω_j and (**b**) with sink in Ω_j. $\mathbf{x}^{(i)}(t)$ and $\mathbf{x}^{(j)}(t)$ represent *real* flows in domains Ω_i and Ω_j, depicted by *thin solid* and *dotted curves*, respectively. $\mathbf{x}_i^{(j)}(t)$ represents *imaginary* flows in domain Ω_i, controlled by the vector fields in Ω_j, which are depicted by *dashed curves*. The flow on the boundary is described by $\mathbf{x}^{(0)}(t)$. The normal and tangential vectors $\mathbf{n}_{\partial\Omega_{ij}}$ and $\mathbf{t}_{\partial\Omega_{ij}}$ on the boundary are depicted. *Hollow circles* are for grazing points on the boundary and filled circles are for starting points $(n_1 + n_2 + 1 = n)$

Theorem 2.11 *For a discontinuous dynamical system in Eq. (2.1), there is a point* $\mathbf{x}^{(0)}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ *at time* t_m *between two adjacent domains* Ω_α $(\alpha = i, j)$. *Suppose* $\mathbf{x}^{(i)}(t_{m+}) = \mathbf{x}_m = \mathbf{x}_i^{(j)}(t_{m\pm})$. *For an arbitrarily small* $\varepsilon > 0$, *there are two time intervals* $[t_{m-\varepsilon}, t_m)$ *and* $[t_{m-\varepsilon}, t_{m+\varepsilon}]$. *A flow* $\mathbf{x}^{(i)}(t)$ *is* $C^{r_i}_{[t_{m-\varepsilon}, t_{m+\varepsilon}]}$*-continuous* $(r_i \geq 2k_i + 1)$ *for time* t

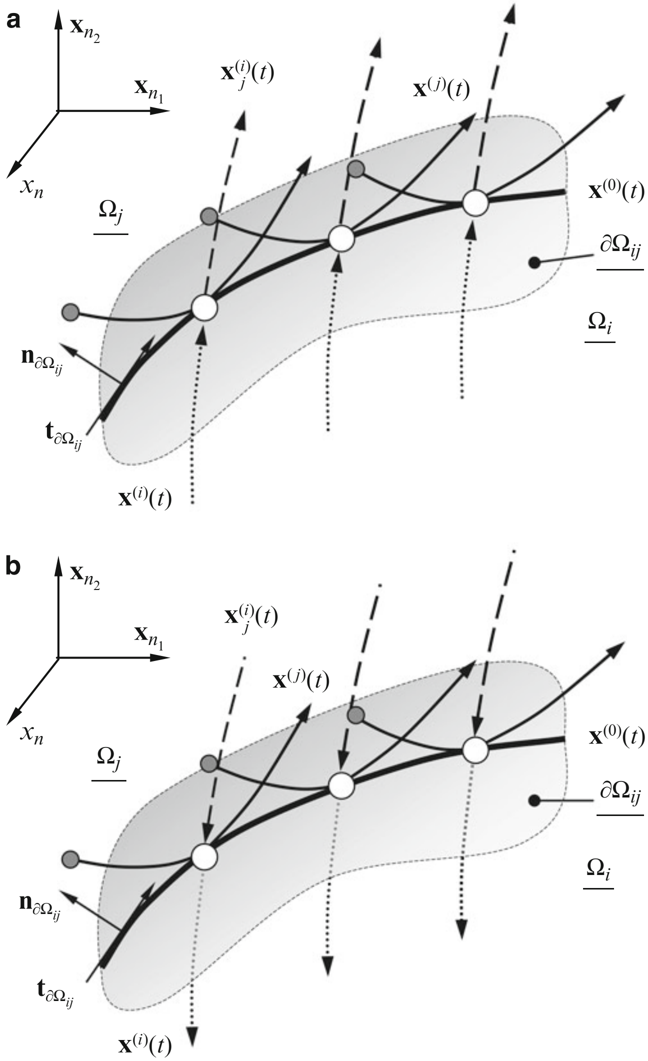

Fig. 2.12 The $(2k_j - 1 : 2k_i)$-tangential flows in Ω_j: (**a**) with sink in Ω_i and (**b**) with source in Ω_i. $\mathbf{x}^{(i)}(t)$ and $\mathbf{x}^{(j)}(t)$ represent the *real* flows in domains Ω_i and Ω_j, depicted by *dotted* and *thin solid curves*, respectively, and $\mathbf{x}_j^{(i)}(t)$ represents *imaginary* flows in domain Ω_j, controlled by the vector fields in Ω_i, which are depicted by *dashed curves*. The flow on the boundary is described by $\mathbf{x}^{(0)}(t)$. The normal and tangential vectors $\mathbf{n}_{\partial\Omega_{ij}}$ and $\mathbf{t}_{\partial\Omega_{ij}}$ on the boundary are depicted. *Hollow circles* are for grazing points on the boundary and *filled circles* are for starting points $(n_1 + n_2 + 1 = n)$

and $||d^{r_i+1}\mathbf{x}^{(i)}/dt^{r_i+1}|| < \infty$, *and a flow* $\mathbf{x}^{(j)}(t)$ *is* $C^{r_j}_{[t_{m-\varepsilon},t_m)}$ *or* $C^{r_j}_{(t_m,t_{m+\varepsilon}]}$*-continuous with* $||d^{r_j+1}\mathbf{x}^{(j)}/dt^{r_j+1}|| < \infty$ $(r_j \geq 2k_j)$*. The flow* $\mathbf{x}^{(i)}(t)$ *of the* $(2k_i-1)$th-*order and* $\mathbf{x}^{(j)}(t)$ *of the* $(2k_j)$th-*order to the boundary* $\partial\Omega_{ij}$ *is* $(2k_i-1:2k_j)$*-tangential flow in domain* Ω_i *if and only if*

$$G^{(s_i,i)}_{\partial\Omega_{ij}}(\mathbf{x}_m,t_{m\pm},\mathbf{p}_i,\boldsymbol{\lambda}) = 0 \text{ for } s_i = 0,1,\cdots,2k_i-2; \tag{2.70}$$

$$G^{(s_j,j)}_{\partial\Omega_{ij}}(\mathbf{x}_m,t_{m\pm},\mathbf{p}_j,\boldsymbol{\lambda}) = 0 \text{ for } s_j = 0,1,\cdots,2k_j-1; \tag{2.71}$$

$$\left.\begin{array}{ll}\text{either} & \left.\begin{array}{l} G^{(2k_i-1,i)}_{\partial\Omega_{ij}}(\mathbf{x}_m,t_{m\pm},\mathbf{p}_i,\boldsymbol{\lambda}) < 0 \\ \left.\begin{array}{l} G^{(2k_j,j)}_{\partial\Omega_{ij}}(\mathbf{x}_m,t_{m-},\mathbf{p}_j,\boldsymbol{\lambda}) < 0 \text{ or} \\ G^{(2k_j,j)}_{\partial\Omega_{ij}}(\mathbf{x}_m,t_{m+},\mathbf{p}_j,\boldsymbol{\lambda}) > 0 \end{array}\right\} \end{array}\right\} \text{for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_j, \\ \text{or} & \left.\begin{array}{l} G^{(2k_i-1,i)}_{\partial\Omega_{ij}}(\mathbf{x}_m,t_{m\pm},\mathbf{p}_i,\boldsymbol{\lambda}) > 0 \\ \left.\begin{array}{l} G^{(2k_j,j)}_{\partial\Omega_{ij}}(\mathbf{x}_m,t_{m-},\mathbf{p}_j,\boldsymbol{\lambda}) > 0 \text{ or} \\ G^{(2k_j,j)}_{\partial\Omega_{ij}}(\mathbf{x}_m,t_{m+},\mathbf{p}_j,\boldsymbol{\lambda}) < 0 \end{array}\right\} \end{array}\right\} \text{for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_i. \end{array}\right. \tag{2.72}$$

Proof The proof is similar to the proof of Theorem 2.2. □

The $(2k_\alpha-1:2k_\beta-1)$-tangential flows in domain Ω_α and Ω_β $(\alpha,\beta \in \{i,j\}$ and $\alpha \neq \beta)$ are sketched in Fig. 2.13 with the corresponding imaginary tangential flows. The real tangential flows are presented by solid curves. Dashed curves denote the imaginary tangential flows. The corresponding definition is given as follows.

Definition 2.22 For a discontinuous dynamical system in Eq. (2.1), there is a point $\mathbf{x}^{(0)}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ at time t_m between two adjacent domains Ω_α $(\alpha = i,j)$. Suppose $\mathbf{x}^{(\alpha)}(t_{m\pm}) = \mathbf{x}_m = \mathbf{x}^{(\beta)}_\alpha(t_{m\pm})$ $(\alpha,\beta \in \{i,j\}$ and $\beta \neq \alpha)$. For an arbitrarily small $\varepsilon > 0$, there is a time interval $[t_{m-\varepsilon},t_{m+\varepsilon}]$. A flow $\mathbf{x}^{(\alpha)}(t)$ is $C^{r_\alpha}_{[t_{m-\varepsilon},t_{m+\varepsilon}]}$-continuous $(r_\alpha \geq 2k_\alpha)$ and $||d^{r_\alpha+1}\mathbf{x}^{(\alpha)}/dt^{r_\alpha+1}|| < \infty$ for time t, and an imaginary flow $\mathbf{x}^{(\beta)}_\alpha(t)$ is $C^{r_\beta}_{[t_{m-\varepsilon},t_{m+\varepsilon}]}$-continuous with $||d^{r_\beta+1}\mathbf{x}^{(\beta)}_\alpha/dt^{r_\beta+1}|| < \infty$ $(r_\beta \geq 2k_\beta)$. The flow $\mathbf{x}^{(\alpha)}(t)$ of the $(2k_\alpha-1)$th-order and $\mathbf{x}^{(\beta)}_\alpha(t)$ of the $(2k_\beta-1)$th-order to the boundary $\partial\Omega_{ij}$ is a $(2k_\alpha-1:2k_\beta-1)$-tangential flow in domain Ω_α if

$$\begin{array}{l} G^{(s_\alpha,\alpha)}_{\partial\Omega_{\alpha\beta}}(\mathbf{x}_m,t_{m\pm},\mathbf{p}_\alpha,\boldsymbol{\lambda}) = 0 \text{ for } s_\alpha = 0,1,\cdots,2k_\alpha-2, \\ G^{(2k_\alpha-1,\alpha)}_{\partial\Omega_{\alpha\beta}}(\mathbf{x}_m,t_{m\pm},\mathbf{p}_\alpha,\boldsymbol{\lambda}) \neq 0; \end{array} \tag{2.73}$$

$$\begin{array}{l} G^{(s_\beta,\beta)}_{\partial\Omega_{\alpha\beta}}(\mathbf{x}_m,t_{m\pm},\mathbf{p}_\beta,\boldsymbol{\lambda}) = 0 \text{ for } s_\beta = 0,1,\cdots,2k_\beta-2, \\ G^{(2k_\beta-1,\beta)}_{\partial\Omega_{\alpha\beta}}(\mathbf{x}_m,t_{m-},\mathbf{p}_\beta,\boldsymbol{\lambda}) \neq 0; \end{array} \tag{2.74}$$

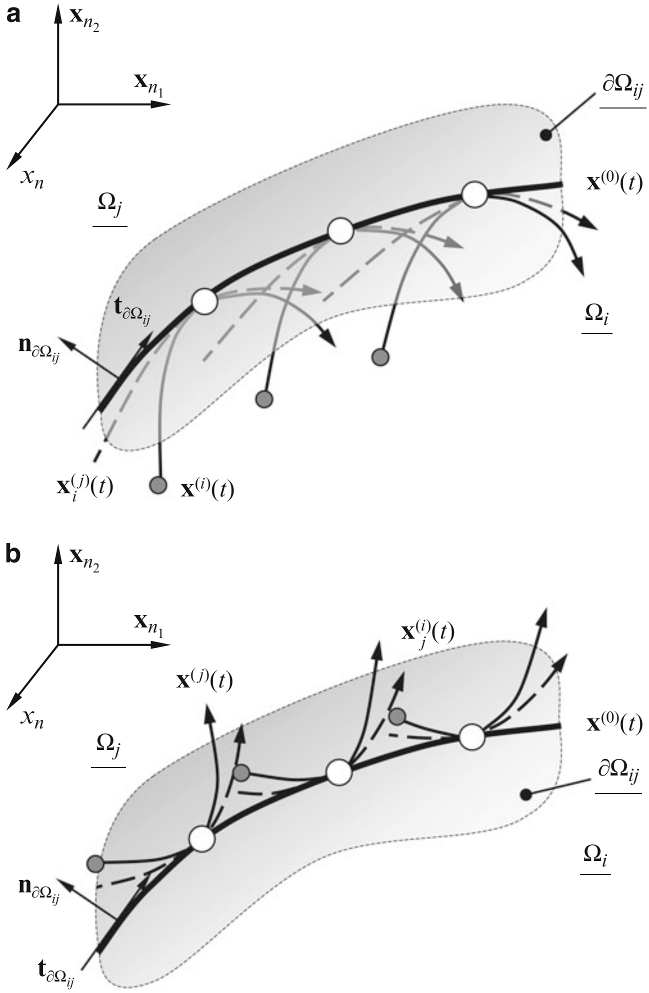

Fig. 2.13 $(2k_i - 1 : 2k_j - 1)$ real and imaginary tangential flows in: (**a**) Ω_i and (**b**) Ω_j. $\mathbf{x}^{(i)}(t)$ and $\mathbf{x}^{(j)}(t)$ represent the *real* flows in domains Ω_i and Ω_j, respectively, which are depicted by the *thin solid curves*. $\mathbf{x}_i^{(j)}(t)$ and $\mathbf{x}_j^{(i)}(t)$ represent the *imaginary* flows in domains Ω_i and Ω_j, respectively, controlled by the vector fields in Ω_j and Ω_i, which are depicted by *dashed curves*. The flow on the boundary is described by $\mathbf{x}^{(0)}(t)$. The normal and tangential vectors $\mathbf{n}_{\partial\Omega_{ij}}$ and $\mathbf{t}_{\partial\Omega_{ij}}$ of the boundary are depicted. *Hollow circles* are for grazing points on the boundary, and *filled circles* are for starting points $(n_1 + n_2 + 1 = n)$

$$\begin{aligned}&\text{either}\quad \left.\begin{aligned}&\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}_{m-\varepsilon})\cdot[\mathbf{x}^{(0)}_{m-\varepsilon}-\mathbf{x}^{(\alpha)}_{m-\varepsilon}]>0\\&\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}_{m+\varepsilon})\cdot[\mathbf{x}^{(\alpha)}_{m+\varepsilon}-\mathbf{x}^{(0)}_{m+\varepsilon}]<0\end{aligned}\right\}\text{for } \mathbf{n}_{\partial\Omega_{\alpha\beta}}\to\Omega_\beta\\&\text{or}\quad \left.\begin{aligned}&\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}_{m-\varepsilon})\cdot[\mathbf{x}^{(0)}_{m-\varepsilon}-\mathbf{x}^{(\alpha)}_{m-\varepsilon}]<0\\&\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}_{m+\varepsilon})\cdot[\mathbf{x}^{(\alpha)}_{m+\varepsilon}-\mathbf{x}^{(0)}_{m+\varepsilon}]>0\end{aligned}\right\}\text{for } \mathbf{n}_{\partial\Omega_{\alpha\beta}}\to\Omega_\alpha,\end{aligned}\tag{2.75}$$

$$\begin{aligned}&\text{either}\quad \left.\begin{aligned}&\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}_{m-\varepsilon})\cdot[\mathbf{x}^{(0)}_{m-\varepsilon}-\mathbf{x}^{(\beta)}_{\alpha(m-\varepsilon)}]>0\\&\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}_{m+\varepsilon})\cdot[\mathbf{x}^{(\beta)}_{\alpha(m+\varepsilon)}-\mathbf{x}^{(0)}_{m+\varepsilon}]<0\end{aligned}\right\}\text{for } \mathbf{n}_{\partial\Omega_{\alpha\beta}}\to\Omega_\beta\\&\text{or}\quad \left.\begin{aligned}&\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}_{m-\varepsilon})\cdot[\mathbf{x}^{(0)}_{m-\varepsilon}-\mathbf{x}^{(\beta)}_{\alpha(m-\varepsilon)}]<0\\&\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}_{m+\varepsilon})\cdot[\mathbf{x}^{(\beta)}_{\alpha(m+\varepsilon)}-\mathbf{x}^{(0)}_{m+\varepsilon}]>0\end{aligned}\right\}\text{for } \mathbf{n}_{\partial\Omega_{\alpha\beta}}\to\Omega_\alpha.\end{aligned}\tag{2.76}$$

The corresponding necessary and sufficient conditions for the tangential flow are given by the following theorem.

Theorem 2.12 *For a discontinuous dynamical system in Eq. (2.1), there is a point* $\mathbf{x}^{(0)}(t_m)\equiv\mathbf{x}_m\in\partial\Omega_{ij}$ *at time* t_m *between two adjacent domains* Ω_α $(\alpha=i,j)$. *Suppose* $\mathbf{x}^{(\alpha)}(t_{m\pm})=\mathbf{x}_m=\mathbf{x}^{(\beta)}_\alpha(t_{m\pm})$ $(\alpha,\beta\in\{i,j\}$ and $\alpha\neq\beta)$. *For an arbitrarily small* $\varepsilon>0$, *there is a time interval* $[t_{m-\varepsilon},t_{m+\varepsilon}]$. *A flow* $\mathbf{x}^{(\alpha)}(t)$ *is* $C^{r_\alpha}_{[t_{m-\varepsilon},t_{m+\varepsilon}]}$*-continuous* $(r_\alpha\geq 2k_\alpha)$ *with* $\|d^{r_\alpha+1}\mathbf{x}^{(\alpha)}/dt^{r_\alpha+1}\|<\infty$ *for time t, and an imaginary flow* $\mathbf{x}^{(\beta)}_\alpha(t)$ *is* $C^{r_\beta}_{[t_{m-\varepsilon},t_{m+\varepsilon}]}$*-continuous* with $\|d^{r_\beta+1}\mathbf{x}^{(\beta)}_\alpha/dt^{r_\beta+1}\|<\infty$ $(r_\beta\geq 2k_\beta)$. *The flow* $\mathbf{x}^{(\alpha)}(t)$ *of the* $(2k_\alpha-1)$th*-order and* $\mathbf{x}^{(\beta)}_\alpha(t)$ *of the* $(2k_\beta-1)$th*-order to the boundary* $\partial\Omega_{ij}$ *is a* $(2k_\alpha-1:2k_\beta-1)$*-tangential flow in domain* Ω_α *if and only if*

$$G^{(s_\alpha,\alpha)}_{\partial\Omega_{\alpha\beta}}(\mathbf{x}_m,t_{m\pm},\mathbf{p}_\alpha,\boldsymbol{\lambda})=0\text{ for } s_\alpha=0,1,\cdots,2k_\alpha-2;\tag{2.77}$$

$$G^{(s_\beta,\beta)}_{\partial\Omega_{\alpha\beta}}(\mathbf{x}_m,t_{m\pm},\mathbf{p}_\beta,\boldsymbol{\lambda})=0\text{ for } s_\beta=0,1,\cdots,2k_\beta-2;\tag{2.78}$$

$$\begin{aligned}&\text{either}\quad \left.\begin{aligned}&G^{(2k_\alpha-1,\alpha)}_{\partial\Omega_{\alpha\beta}}(\mathbf{x}_m,t_{m\pm},\mathbf{p}_\alpha,\boldsymbol{\lambda})<0\\&G^{(2k_\beta-1,\beta)}_{\partial\Omega_{\alpha\beta}}(\mathbf{x}_m,t_{m\pm},\mathbf{p}_\beta,\boldsymbol{\lambda})<0\end{aligned}\right\}\text{for } \mathbf{n}_{\partial\Omega_{\alpha\beta}}\to\Omega_\beta\\&\text{or}\quad \left.\begin{aligned}&G^{(2k_\alpha-1,\alpha)}_{\partial\Omega_{\alpha\beta}}(\mathbf{x}_m,t_{m\pm},\mathbf{p}_\alpha,\boldsymbol{\lambda})>0\\&G^{(2k_\beta-1,\beta)}_{\partial\Omega_{\alpha\beta}}(\mathbf{x}_m,t_{m\pm},\mathbf{p}_\beta,\boldsymbol{\lambda})>0\end{aligned}\right\}\text{for } \mathbf{n}_{\partial\Omega_{\alpha\beta}}\to\Omega_\alpha.\end{aligned}\tag{2.79}$$

Proof The proof is similar to the proof of Theorem 2.2. □

The $(2k_\alpha-1:2k_\beta-1)$-double tangential flows are sketched in Fig. 2.14a by the solid curves. The double tangential flow is formed by the two real tangential flows

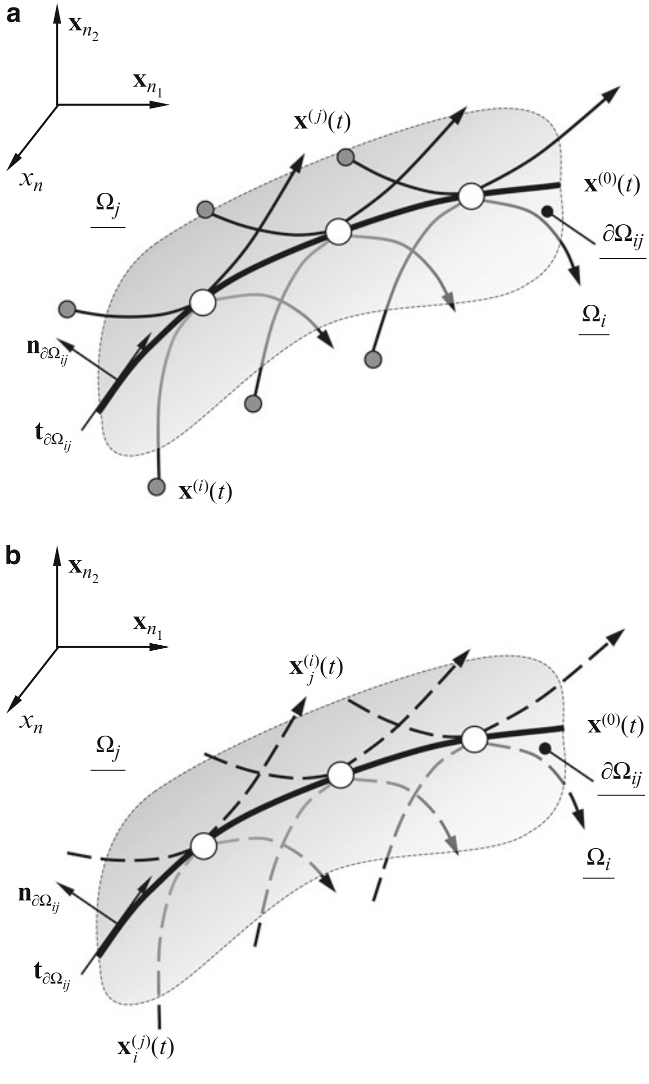

Fig. 2.14 (**a**) $(2k_i - 1 : 2k_j - 1)$-double tangential flow in both Ω_i and Ω_j and (**b**) $(2k_i - 1 : 2k_j - 1)$-double inaccessible tangential flow in both Ω_i and Ω_j. $\mathbf{x}^{(i)}(t)$ and $\mathbf{x}^{(j)}(t)$ represent the *real* flows in domains Ω_i and Ω_j, respectively, which are depicted by *thin solid curves*. $\mathbf{x}_i^{(j)}(t)$ and $\mathbf{x}_j^{(i)}(t)$ represent the *imaginary* flows in domains Ω_i and Ω_j, respectively, controlled by the vector fields in Ω_j and Ω_i, which are depicted by *dashed curves*. The flow on the boundary is described by $\mathbf{x}^{(0)}(t)$. The normal and tangential vectors $\mathbf{n}_{\partial\Omega_{ij}}$ and $\mathbf{t}_{\partial\Omega_{ij}}$ of the boundary are depicted. *Hollow circles* are for grazing points on the boundary and *filled circles* are for starting points $(n_1 + n_2 + 1 = n)$

in both domains. The $(2k_\alpha - 1 : 2k_\beta - 1)$-double inaccessible tangential flows are sketched in Fig. 2.14b by the dashed curves. Such a double inaccessible flow is formed by two imaginary tangential flows to the boundary. No any flows in the two domains can access the boundary.

Definition 2.23 For a discontinuous dynamical system in Eq. (2.1), there is a point $\mathbf{x}^{(0)}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ at time t_m between two adjacent domains Ω_α ($\alpha = i, j$). Suppose $\mathbf{x}^{(\alpha)}(t_{m\pm}) = \mathbf{x}_m = \mathbf{x}^{(\beta)}(t_{m\pm})$ ($\alpha, \beta \in \{i, j\}$ and $\alpha \neq \beta$). For an arbitrarily small $\varepsilon > 0$, there is a time interval $[t_{m-\varepsilon}, t_{m+\varepsilon}]$. A flow $\mathbf{x}^{(\alpha)}(t)$ is $C^{r_\alpha}_{[t_{m-\varepsilon}, t_{m+\varepsilon}]}$-continuous ($r_\alpha \geq 2k_\alpha$) for time t with $\|d^{r_\alpha+1}\mathbf{x}^{(\alpha)}/dt^{r_\alpha+1}\| < \infty$, and the flow $\mathbf{x}^{(\beta)}(t)$ is $C^{r_\beta}_{[t_{m-\varepsilon}, t_{m+\varepsilon}]}$-continuous ($r_\beta \geq 2k_\beta$) with $\|d^{r_\beta+1}\mathbf{x}^{(\beta)}/dt^{r_\beta+1}\| < \infty$. The flow $\mathbf{x}^{(\alpha)}(t)$ of the $(2k_\alpha - 1)$th-order and $\mathbf{x}^{(\beta)}(t)$ of the $(2k_\beta - 1)$th-order to the boundary $\partial\Omega_{\alpha\beta}$ is a $(2k_\alpha - 1 : 2k_\beta - 1)$-double tangential flow if

$$
\left.
\begin{aligned}
&G^{(s_\alpha,\alpha)}_{\partial\Omega_{\alpha\beta}}(\mathbf{x}_m, t_{m\pm}, \mathbf{p}_\alpha, \boldsymbol{\lambda}) = 0 \text{ for } s_\alpha = 0, 1, \cdots, 2k_\alpha - 2,\\
&G^{(2k_\alpha-1,\alpha)}_{\partial\Omega_{\alpha\beta}}(\mathbf{x}_m, t_{m\pm}, \mathbf{p}_\alpha, \boldsymbol{\lambda}) \neq 0;
\end{aligned}
\right. \tag{2.80}
$$

$$
\left.
\begin{aligned}
&G^{(s_\beta,\beta)}_{\partial\Omega_{\alpha\beta}}(\mathbf{x}_m, t_{m\pm}, \mathbf{p}_\beta, \boldsymbol{\lambda}) = 0 \text{ for } s_\beta = 0, 1, \cdots, 2k_\beta - 2,\\
&G^{(2k_\beta-1,\beta)}_{\partial\Omega_{\alpha\beta}}(\mathbf{x}_m, t_{m-}, \mathbf{p}_\beta, \boldsymbol{\lambda}) \neq 0;
\end{aligned}
\right. \tag{2.81}
$$

$$
\begin{aligned}
&\text{either} \quad \left.
\begin{aligned}
&\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}_{m-\varepsilon}) \cdot [\mathbf{x}^{(0)}_{m-\varepsilon} - \mathbf{x}^{(\alpha)}_{m-\varepsilon}] > 0\\
&\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}_{m+\varepsilon}) \cdot [\mathbf{x}^{(\alpha)}_{m+\varepsilon} - \mathbf{x}^{(0)}_{m+\varepsilon}] < 0
\end{aligned}
\right\} \text{ for } \mathbf{n}_{\partial\Omega_{\alpha\beta}} \to \Omega_\beta\\
&\text{or} \quad \left.
\begin{aligned}
&\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}_{m-\varepsilon}) \cdot [\mathbf{x}^{(0)}_{m-\varepsilon} - \mathbf{x}^{(\alpha)}_{m-\varepsilon}] < 0\\
&\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}_{m+\varepsilon}) \cdot [\mathbf{x}^{(\alpha)}_{m+\varepsilon} - \mathbf{x}^{(0)}_{m+\varepsilon}] > 0
\end{aligned}
\right\} \text{ for } \mathbf{n}_{\partial\Omega_{\alpha\beta}} \to \Omega_\alpha;
\end{aligned} \tag{2.82}
$$

$$
\begin{aligned}
&\text{either} \quad \left.
\begin{aligned}
&\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}_{m-\varepsilon}) \cdot [\mathbf{x}^{(0)}_{m-\varepsilon} - \mathbf{x}^{(\beta)}_{m-\varepsilon}] < 0\\
&\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}_{m+\varepsilon}) \cdot [\mathbf{x}^{(\beta)}_{m+\varepsilon} - \mathbf{x}^{(0)}_{m+\varepsilon}] > 0
\end{aligned}
\right\} \text{ for } \mathbf{n}_{\partial\Omega_{\alpha\beta}} \to \Omega_\beta\\
&\text{or} \quad \left.
\begin{aligned}
&\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}_{m-\varepsilon}) \cdot [\mathbf{x}^{(0)}_{m-\varepsilon} - \mathbf{x}^{(\beta)}_{m-\varepsilon}] > 0\\
&\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}_{m+\varepsilon}) \cdot [\mathbf{x}^{(\beta)}_{m+\varepsilon} - \mathbf{x}^{(0)}_{m+\varepsilon}] < 0
\end{aligned}
\right\} \text{ for } \mathbf{n}_{\partial\Omega_{\alpha\beta}} \to \Omega_\alpha.
\end{aligned} \tag{2.83}
$$

The corresponding necessary and sufficient conditions for the tangential flows are given through the following theorem.

Theorem 2.13 *For a discontinuous dynamical system in Eq. (2.1), there is a point* $\mathbf{x}^{(0)}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ *at time* t_m *between two adjacent domains* Ω_α *(*$\alpha = i, j$*). Suppose* $\mathbf{x}^{(\alpha)}(t_{m\pm}) = \mathbf{x}_m = \mathbf{x}^{(\beta)}(t_{m\pm})$ *(*$\alpha, \beta \in \{i, j\}$ *and* $\beta \neq \alpha$*). For an arbitrarily small* $\varepsilon > 0$*, there is a time interval* $[t_{m-\varepsilon}, t_{m+\varepsilon}]$*. A flow* $\mathbf{x}^{(\alpha)}(t)$ *is* $C^{r_\alpha}_{[t_{m-\varepsilon}, t_{m+\varepsilon}]}$*-continuous (*$r_\alpha \geq 2k_\alpha$*) and* $\|d^{r_\alpha+1}\mathbf{x}^{(\alpha)}/dt^{r_\alpha+1}\| < \infty$ *for time t, and a flow* $\mathbf{x}^{(\beta)}(t)$ *is* $C^{r_\beta}_{[t_{m-\varepsilon}, t_{m+\varepsilon}]}$*-continuous (*$r_\beta \geq 2k_\beta$*) and* $\|d^{r_\beta+1}\mathbf{x}^{(\beta)}/dt^{r_\beta+1}\| < \infty$*. The flow* $\mathbf{x}^{(\alpha)}(t)$ *of the* $(2k_\alpha - 1)$th-*order and*

$\mathbf{x}^{(\beta)}(t)$ *of the* $(2k_\beta - 1)$th-*order to the boundary* $\partial\Omega_{\alpha\beta}$ *is a* $(2k_\alpha - 1 : 2k_\beta - 1)$-*double tangential flow if and only if*

$$G^{(s_\alpha,\alpha)}_{\partial\Omega_{\alpha\beta}}(\mathbf{x}_m, t_{m\pm}, \mathbf{p}_\alpha, \boldsymbol{\lambda}) = 0 \text{ for } s_\alpha = 0, 1, \cdots, 2k_\alpha - 2; \tag{2.84}$$

$$G^{(s_\beta,\beta)}_{\partial\Omega_{\alpha\beta}}(\mathbf{x}_m, t_{m\pm}, \mathbf{p}_\beta, \boldsymbol{\lambda}) = 0 \text{ for } s_\beta = 0, 1, \cdots, 2k_\beta - 2; \tag{2.85}$$

$$\begin{aligned} &\text{either} && \left.\begin{aligned} G^{(2k_\alpha-1,\alpha)}_{\partial\Omega_{\alpha\beta}}(\mathbf{x}_m, t_{m\pm}, \mathbf{p}_\alpha, \boldsymbol{\lambda}) < 0 \\ G^{(2k_\beta-1,\beta)}_{\partial\Omega_{\alpha\beta}}(\mathbf{x}_m, t_{m\pm}, \mathbf{p}_\beta, \boldsymbol{\lambda}) > 0 \end{aligned}\right\} \text{for } \mathbf{n}_{\partial\Omega_{\alpha\beta}} \to \Omega_\beta \\ &\text{or} && \left.\begin{aligned} G^{(2k_\alpha-1,\alpha)}_{\partial\Omega_{\alpha\beta}}(\mathbf{x}_m, t_{m\pm}, \mathbf{p}_\alpha, \boldsymbol{\lambda}) > 0 \\ G^{(2k_\beta-1,\beta)}_{\partial\Omega_{\alpha\beta}}(\mathbf{x}_m, t_{m\pm}, \mathbf{p}_\beta, \boldsymbol{\lambda}) < 0 \end{aligned}\right\} \text{for } \mathbf{n}_{\partial\Omega_{\alpha\beta}} \to \Omega_\alpha. \end{aligned} \tag{2.86}$$

Proof The proof is similar to the proof of Theorem 2.2. □

Definition 2.24 For a discontinuous dynamical system in Eq. (2.1), there is a point $\mathbf{x}^{(0)}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ at time t_m between two adjacent domains Ω_α $(\alpha = i, j)$. Suppose $\mathbf{x}^{(\alpha)}_\beta(t_{m\pm}) = \mathbf{x}_m = \mathbf{x}^{(\beta)}_\alpha(t_{m\pm})$ ($\alpha, \beta \in \{i, j\}$ and $\beta \neq \alpha$). For an arbitrarily small $\varepsilon > 0$, there is a time interval $[t_{m-\varepsilon}, t_{m+\varepsilon}]$. An imaginary flow $\mathbf{x}^{(\alpha)}_\beta(t)$ is $C^{r_\alpha}_{[t_{m-\varepsilon}, t_{m+\varepsilon}]}$-continuous $(r_\alpha \geq 2k_\alpha)$ for time t and $||d^{r_\alpha+1}\mathbf{x}^{(\alpha)}_\beta/dt^{r_\alpha+1}|| < \infty$, and an imaginary flow $\mathbf{x}^{(\beta)}_\alpha(t)$ is $C^{r_\beta}_{[t_{m-\varepsilon}, t_{m+\varepsilon}]}$-continuous ($r_\beta \geq 2k_\beta$) for time t with $||d^{r_\beta+1}\mathbf{x}^{(\beta)}_\alpha/dt^{r_\beta+1}|| < \infty$. The imaginary flow $\mathbf{x}^{(\alpha)}_\beta(t)$ of the $(2k_\alpha - 1)$th-order and the imaginary $\mathbf{x}^{(\beta)}_\alpha(t)$ of the $(2k_\beta - 1)$th-order to the boundary $\partial\Omega_{ij}$ is a $(2k_\alpha - 1 : 2k_\beta - 1)$-double inaccessible tangential flow if

$$\begin{aligned} &G^{(s_\alpha,\alpha)}_{\partial\Omega_{\alpha\beta}}(\mathbf{x}_m, t_{m\pm}, \mathbf{p}_\alpha, \boldsymbol{\lambda}) = 0 \text{ for } s_\alpha = 0, 1, \cdots, 2k_\alpha - 2, \\ &G^{(2k_\alpha-1,\alpha)}_{\partial\Omega_{\alpha\beta}}(\mathbf{x}_m, t_{m\pm}, \mathbf{p}_\alpha, \boldsymbol{\lambda}) \neq 0; \end{aligned} \tag{2.87}$$

$$\begin{aligned} &G^{(s_\beta,\beta)}_{\partial\Omega_{\alpha\beta}}(\mathbf{x}_m, t_{m\pm}, \mathbf{p}_\beta, \boldsymbol{\lambda}) = 0 \text{ for } s_\beta = 0, 1, \cdots, 2k_\beta - 2, \\ &G^{(2k_\beta-1,\beta)}_{\partial\Omega_{\alpha\beta}}(\mathbf{x}_m, t_{m-}, \mathbf{p}_\beta, \boldsymbol{\lambda}) \neq 0; \end{aligned} \tag{2.88}$$

$$\begin{aligned} &\text{either} && \left.\begin{aligned} \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}_{m-\varepsilon}) \cdot [\mathbf{x}^{(0)}_{m-\varepsilon} - \mathbf{x}^{(\alpha)}_{\beta(m-\varepsilon)}] < 0 \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}_{m+\varepsilon}) \cdot [\mathbf{x}^{(\alpha)}_{\beta(m+\varepsilon)} - \mathbf{x}^{(0)}_{m+\varepsilon}] > 0 \end{aligned}\right\} \text{for } \mathbf{n}_{\partial\Omega_{\alpha\beta}} \to \Omega_\beta \\ &\text{or} && \left.\begin{aligned} \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}_{m-\varepsilon}) \cdot [\mathbf{x}^{(0)}_{m-\varepsilon} - \mathbf{x}^{(\alpha)}_{\beta(m-\varepsilon)}] > 0 \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}_{m+\varepsilon}) \cdot [\mathbf{x}^{(\alpha)}_{\beta(m+\varepsilon)} - \mathbf{x}^{(0)}_{m+\varepsilon}] < 0 \end{aligned}\right\} \text{for } \mathbf{n}_{\partial\Omega_{\alpha\beta}} \to \Omega_\alpha, \end{aligned} \tag{2.89}$$

$$\begin{array}{ll}\text{either} & \left.\begin{array}{l}\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}_{m-\varepsilon})\cdot[\mathbf{x}^{(0)}_{m-\varepsilon}-\mathbf{x}^{(\beta)}_{\alpha(m-\varepsilon)}]>0\\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}_{m+\varepsilon})\cdot[\mathbf{x}^{(\beta)}_{\alpha(m+\varepsilon)}-\mathbf{x}^{(0)}_{m+\varepsilon}]<0\end{array}\right\}\text{for } \mathbf{n}_{\partial\Omega_{\alpha\beta}}\to\Omega_{\beta}\\ \text{or} & \left.\begin{array}{l}\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}_{m-\varepsilon})\cdot[\mathbf{x}^{(0)}_{m-\varepsilon}-\mathbf{x}^{(\beta)}_{\alpha(m-\varepsilon)}]<0\\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}_{m+\varepsilon})\cdot[\mathbf{x}^{(\beta)}_{\alpha(m+\varepsilon)}-\mathbf{x}^{(0)}_{m+\varepsilon}]>0\end{array}\right\}\text{for } \mathbf{n}_{\partial\Omega_{\alpha\beta}}\to\Omega_{\alpha}.\end{array} \tag{2.90}$$

The necessary and sufficient conditions for the tangential flows are given.

Theorem 2.14 *For a discontinuous dynamical system in Eq. (2.1), there is a point* $\mathbf{x}^{(0)}(t_m)\equiv\mathbf{x}_m\in\partial\Omega_{ij}$ *at time* t_m *between two adjacent domains* Ω_α $(\alpha=i,j)$*. Suppose* $\mathbf{x}^{(\alpha)}_{\beta}(t_{m\pm})=\mathbf{x}_m=\mathbf{x}^{(\beta)}_{\alpha}(t_{m\pm})$ $(\alpha,\beta\in\{i,j\}$ *and* $\beta\neq\alpha)$*. For an arbitrarily small* $\varepsilon>0$*, there is a time interval* $[t_{m-\varepsilon},t_{m+\varepsilon}]$*. An imaginary flow* $\mathbf{x}^{(\alpha)}_{\beta}(t)$ *is* $C^{r_\alpha}_{[t_{m-\varepsilon},t_{m+\varepsilon}]}$*-continuous* $(r_\alpha\geq 2k_\alpha)$ *for time t with* $||d^{r_\alpha+1}\mathbf{x}^{(\alpha)}_{\beta}/dt^{r_\alpha+1}||<\infty$*, and an imaginary flow* $\mathbf{x}^{(\beta)}_{\alpha}(t)$ *is* $C^{r_\beta}_{[t_{m-\varepsilon},t_{m+\varepsilon}]}$ *-continuous* ($r_\beta\geq 2k_\beta$) for time *t with* $||d^{r_\beta+1}\mathbf{x}^{(\beta)}_{\alpha}/dt^{r_\beta+1}||<\infty$ *. The imaginary flow* $\mathbf{x}^{(\alpha)}_{\beta}(t)$ *of the* $(2k_\alpha-1)$th*-order and the imaginary* $\mathbf{x}^{(\beta)}_{\alpha}(t)$ *of the* $(2k_\beta-1)$th*-order to the boundary* $\partial\Omega_{ij}$ *is a* $(2k_\alpha-1:2k_\beta-1)$*-double inaccessible tangential flow if and only if*

$$G^{(s_\alpha,\alpha)}_{\partial\Omega_{\alpha\beta}}(\mathbf{x}_m,t_{m\pm},\mathbf{p}_\alpha,\boldsymbol{\lambda})=0 \text{ for } s_\alpha=0,1,\cdots,2k_\alpha-2; \tag{2.91}$$

$$G^{(s_\beta,\beta)}_{\partial\Omega_{\alpha\beta}}(\mathbf{x}_m,t_{m\pm},\mathbf{p}_\beta,\boldsymbol{\lambda})=0 \text{ for } s_\beta=0,1,\cdots,2k_\beta-2; \tag{2.92}$$

$$\begin{array}{ll}\text{either} & \left.\begin{array}{l}G^{(2k_\alpha-1,\alpha)}_{\partial\Omega_{\alpha\beta}}(\mathbf{x}_m,t_{m\pm},\mathbf{p}_\alpha,\boldsymbol{\lambda})>0\\ G^{(2k_\beta-1,\beta)}_{\partial\Omega_{\alpha\beta}}(\mathbf{x}_m,t_{m\pm},\mathbf{p}_\beta,\boldsymbol{\lambda})<0\end{array}\right\}\text{for } \mathbf{n}_{\partial\Omega_{\alpha\beta}}\to\Omega_{\beta}\\ \text{or} & \left.\begin{array}{l}G^{(2k_\alpha-1,\alpha)}_{\partial\Omega_{\alpha\beta}}(\mathbf{x}_m,t_{m\pm},\mathbf{p}_\alpha,\boldsymbol{\lambda})<0\\ G^{(2k_\beta-1,\beta)}_{\partial\Omega_{\alpha\beta}}(\mathbf{x}_m,t_{m\pm},\mathbf{p}_\beta,\boldsymbol{\lambda})>0\end{array}\right\}\text{for } \mathbf{n}_{\partial\Omega_{\alpha\beta}}\to\Omega_{\alpha}.\end{array} \tag{2.93}$$

Proof The proof is similar to the proof of Theorem 2.2. □

2.6 Flow Switching Bifurcations

In this section, the flow switching bifurcations from the passable to non-passable flow and the sliding fragmentation bifurcation from the non-passable to passable flow will be discussed. Before discussion of switching bifurcations, the $(m_i:m_j)$ product of the G-functions on the boundary $\partial\Omega_{ij}$ is introduced.

Definition 2.25 For a discontinuous dynamical system in Eq. (2.1), there is a point $\mathbf{x}^{(0)}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ at time t_m between two adjacent domains Ω_α $(\alpha = i,j)$. Suppose $\mathbf{x}^{(i)}(t_{m\pm}) = \mathbf{x}_m = \mathbf{x}^{(j)}(t_{m\mp})$. For an arbitrarily small $\varepsilon > 0$, there is a time interval $[t_{m-\varepsilon}, t_{m+\varepsilon}]$. A flow $\mathbf{x}^{(i)}(t)$ is $C^{r_i}_{[t_{m-\varepsilon},t_{m+\varepsilon}]}$-continuous time t with $||d^{r_i+1}\mathbf{x}^{(i)}/dt^{r_i+1}|| < \infty$ $(r_i \geq m_i + 1)$, and a flow $\mathbf{x}^{(j)}(t)$ is $C^{r_j}_{[t_{m-\varepsilon},t_{m+\varepsilon}]}$-continuous $(r_j \geq m_j + 1)$ with $||d^{r_j+1}\mathbf{x}^{(j)}/dt^{r_j+1}|| < \infty$. *The* $(m_i : m_j)$*-product of G-functions* on the boundary $\partial\Omega_{ij}$ is defined as

$$
\begin{aligned}
L_{ij}^{(m_i:m_j)}(t_m) &\equiv L_{ij}^{(m_i:m_j)}(\mathbf{x}_m, t_m, \mathbf{p}_i, \mathbf{p}_j, \boldsymbol{\lambda}) \\
&= G_{\partial\Omega_{ij}}^{(m_i,i)}(\mathbf{x}_m, t_{m-}, \mathbf{p}_i, \boldsymbol{\lambda}) \times G_{\partial\Omega_{ij}}^{(m_j,j)}(\mathbf{x}_m, t_{m+}, \mathbf{p}_j, \boldsymbol{\lambda})
\end{aligned} \tag{2.94}
$$

and for $m_i = m_j = 0$, we have $L_{ij}^{(0:0)} = L_{ij}$

$$
\begin{aligned}
L_{ij}(t_m) &\equiv L_{ij}(\mathbf{x}_m, t_m, \mathbf{p}_i, \mathbf{p}_j, \boldsymbol{\lambda}) \\
&= G_{\partial\Omega_{ij}}^{(i)}(\mathbf{x}_m, t_{m-}, \mathbf{p}_i, \boldsymbol{\lambda}) \times G_{\partial\Omega_{ij}}^{(j)}(\mathbf{x}_m, t_{m+}, \mathbf{p}_j, \boldsymbol{\lambda}).
\end{aligned} \tag{2.95}
$$

From the foregoing definition, the products of G-functions for the full passable, sink, and source on the boundary $\partial\Omega_{\alpha\beta}$ are

$$
\left.
\begin{aligned}
&L_{\alpha\beta}^{(2k_\alpha:2k_\beta)}(t_m) > 0 \text{ on } \overrightarrow{\partial\Omega}_{\alpha\beta}; \\
&L_{\alpha\beta}^{(2k_\alpha:2k_\beta)}(t_m) < 0 \text{ on } \overline{\partial\Omega}_{\alpha\beta} = \widetilde{\partial\Omega}_{\alpha\beta} \cup \widehat{\partial\Omega}_{\alpha\beta}.
\end{aligned}
\right\} \tag{2.96}
$$

where $\overrightarrow{\partial\Omega}_{\alpha\beta}$, $\widetilde{\partial\Omega}_{\alpha\beta}$ and $\widehat{\partial\Omega}_{\alpha\beta}$ are passable, sink, and source boundaries, respectively. $\overline{\partial\Omega}_{\alpha\beta}$ is the non-passable boundary, including sink and source boundaries. Such boundaries are relative to the passable, sink, and source flows at the boundaries in discontinuous dynamical systems. The switching bifurcation of a flow at $(t_m, \mathbf{x}_m)$ on the boundary $\partial\Omega_{\alpha\beta}$ requires

$$
L_{\alpha\beta}^{(2k_\alpha:2k_\beta)}(t_m) = 0. \tag{2.97}
$$

For a passable flow at $\mathbf{x}(t_m) \equiv \mathbf{x}_m \in [\mathbf{x}_{m_1}, \mathbf{x}_{m_2}] \subset \overrightarrow{\partial\Omega}_{ij}$, consider a time interval $[t_{m_1}, t_{m_2}]$ for $[\mathbf{x}_{m_1}, \mathbf{x}_{m_2}]$ on the boundary and the product of G-functions for $t_m \in [t_{m_1}, t_{m_2}]$ and $\mathbf{x}_m \in [\mathbf{x}_{m_1}, \mathbf{x}_{m_2}]$ is positive, i.e., $L_{ij}^{(2k_i:2k_j)}(t_m) > 0$. To determine the switching bifurcation, the global minimum of such a product of G-functions should be determined. Because $\mathbf{x}_m$ is a function of t_m, the two total derivatives of $L_{ij}^{(2k_i:2k_j)}(t_m)$ are introduced by

$$
\begin{aligned}
DL_{ij}^{(2k_i:2k_j)} &= \nabla L_{ij}^{(2k_i:2k_j)}(\mathbf{x}_m, t_m, \mathbf{p}_i, \mathbf{p}_j, \boldsymbol{\lambda}) \cdot \mathbf{F}_{ij}^{(0)}(\mathbf{x}_m, t_m) \\
&\quad + \partial_{t_m} L_{ij}^{(2k_i:2k_j)}(\mathbf{x}_m, t_m, \mathbf{p}_i, \mathbf{p}_j, \boldsymbol{\lambda}), \\
D^r L_{ij}^{(2k_i:2k_j)} &= D^{r-1}\{DL_{ij}^{(2k_i:2k_j)}(\mathbf{x}_m, t_m, \mathbf{p}_i, \mathbf{p}_j, \boldsymbol{\lambda})\}
\end{aligned} \tag{2.98}
$$

for $r = 1, 2, \cdots$. Thus, the local minimum of $L_{ij}^{(2k_i:2k_j)}(t_m)$ is determined by

$$D^r L_{ij}^{(2k_i:2k_j)}(t_m) = 0, (r = 1, 2, \cdots, 2l - 1) \tag{2.99}$$

$$D^{2l} L_{ij}^{(2k_i:2k_j)}(t_m) > 0. \tag{2.100}$$

Definition 2.26 For a discontinuous dynamical system in Eq. (2.1), there is a point $\mathbf{x}^{(0)}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ at time t_m between two adjacent domains Ω_α $(\alpha = i, j)$. Suppose $\mathbf{x}^{(i)}(t_{m\pm}) = \mathbf{x}_m = \mathbf{x}^{(j)}(t_{m\mp})$. For an arbitrarily small $\varepsilon > 0$, there are two time intervals (i.e., $[t_{m-\varepsilon}, t_m)$ and $(t_m, t_{m+\varepsilon}]$). A flow $\mathbf{x}^{(i)}(t)$ is $C^{r_i}_{[t_{m-\varepsilon}, t_m)}$-continuous $(r_i \geq 2k_i + 1)$ with $||d^{r_i+1}\mathbf{x}^{(i)}/dt^{r_i+1}|| < \infty$ for time t, and a flow $\mathbf{x}^{(j)}(t)$ is $C^{r_j}_{(t_m, t_{m+\varepsilon}]}$-continuous $(r_j \geq 2k_j + 1)$ with $||d^{r_j+1}\mathbf{x}^{(j)}/dt^{r_j+1}|| < \infty$. The local minimum value set of the $(2k_i : 2k_j)$-product of G-functions (i.e., $L_{ij}^{(2k_i:2k_j)}(t_m)$) is defined by

$$_{\min}L_{ij}^{(2k_i:2k_j)}(t_m) = \left\{ L_{ij}^{(2k_i:2k_j)}(t_m) \,\middle|\, \begin{array}{l} \text{for } t_m \in [t_{m_1}, t_{m_2}] \text{ and } \mathbf{x}_m \in [\mathbf{x}_{m_1}, \mathbf{x}_{m_2}], \\ D^r L_{ij}^{(2k_i:2k_j)} = 0 \text{ for } r = \{1, 2, \cdots 2l - 1\}, \\ \text{and } D^{2l} L_{ij}^{(2k_i:2k_j)} > 0. \end{array} \right\} \tag{2.101}$$

From the local minimum set of $L_{ij}^{(2k_i:2k_j)}(t_m)$, the global minimum values of $L_{ij}^{(2k_i:2k_j)}(t_m)$ is defined as follows.

Definition 2.27 For a discontinuous dynamical system in Eq. (2.1), there is a point $\mathbf{x}^{(0)}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ at time t_m between two adjacent domains Ω_α $(\alpha = i, j)$. Suppose $\mathbf{x}^{(i)}(t_{m\pm}) = \mathbf{x}_m = \mathbf{x}^{(j)}(t_{m\mp})$. For an arbitrarily small $\varepsilon > 0$, there are two time intervals (i.e.,$[t_{m-\varepsilon}, t_m)$ and $(t_m, t_{m+\varepsilon}]$). A flow $\mathbf{x}^{(i)}(t)$ is $C^{r_i}_{[t_{m-\varepsilon}, t_m)}$-continuous $(r_i \geq 2k_i + 1)$ with $||d^{r_i+1}\mathbf{x}^{(i)}/dt^{r_i+1}|| < \infty$ for time t, and a flow $\mathbf{x}^{(j)}(t)$ is $C^{r_j}_{(t_m, t_{m+\varepsilon}]}$-continuous $(r_j \geq 2k_j + 1)$ with $||d^{r_j+1}\mathbf{x}^{(j)}/dt^{r_j+1}|| < \infty$. The global minimum value of *the* $(2k_i : 2k_j)$*-product of* G-functions (i.e., $L_{ij}^{(2k_i:2k_j)}(t_m)$) is defined by

$$_{G\min}L_{ij}^{(2k_i:2k_j)}(t_m) = \min_{t_m \in [t_{m_1}, t_{m_2}]} \left\{ {_{\min}L_{ij}^{(2k_i:2k_j)}(t_m)}, L_{ij}^{(2k_i:2k_j)}(t_{m_1}), L_{ij}^{(2k_i:2k_j)}(t_{m_2}) \right\} \tag{2.102}$$

To consider the switching bifurcation varying with the system parameter $\mathbf{q} \in \{\mathbf{p}_i, \mathbf{p}_j, \boldsymbol{\lambda}\}$, $D^r L_{ij}^{(2k_j:2k_j)}$ in Eq. (2.98) is replaced by $d^r L_{ij}^{(2k_i:2k_j)}/d\mathbf{q}^r$. Similarly, the maximum set of the $(2k_i : 2k_j)$-product of G-functions (i.e., $L_{ij}^{(2k_i:2k_j)}(t_m)$) can be developed as follows.

Definition 2.28 For a discontinuous dynamical system in Eq. (2.1), there is a point $\mathbf{x}^{(0)}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ at time t_m between two adjacent domains Ω_α $(\alpha = i, j)$.

Suppose $\mathbf{x}^{(\alpha)}(t_{m\pm}) = \mathbf{x}_m$ $(\alpha \in \{i,j\})$. For an arbitrarily small $\varepsilon > 0$, there are two time intervals ($[t_{m-\varepsilon}, t_m)$ or $(t_m, t_{m+\varepsilon}]$). The flow $\mathbf{x}^{(i)}(t)$ is $C^{r_i}_{[t_{m-\varepsilon},t_m)}$ or $C^{r_i}_{[t_{m-\varepsilon},t_{m+\varepsilon}]}$-continuous $(r_i \geq 2k_i + 1)$ for time t and $||d^{r_i+1}\mathbf{x}^{(i)}/dt^{r_i+1}|| < \infty$. The flow $\mathbf{x}^{(j)}(t)$ is $C^{r_j}_{[t_{m-\varepsilon},t_{m+\varepsilon}]}$ or $C^{r_j}_{(t_m,t_{m+\varepsilon}]}$-continuous for time t and $||d^{r_j+1}\mathbf{x}^{(j)}/dt^{r_j+1}|| < \infty$ $(r_j \geq 2k_j + 1)$. *The local maximum set* of the $(2k_i : 2k_j)$ product of G-functions (i.e., $L_{ij}^{(2k_i:2k_j)}(t_m)$) is defined by

$$
{}_{\max}L_{ij}^{(2k_i:2k_j)}(t_m) = \left\{ L_{ij}^{(2k_i:2k_j)}(t_m) \left| \begin{array}{l} \text{for } t_m \in [t_{m_1}, t_{m_2}] \text{ and } \mathbf{x}_m \in [\mathbf{x}_{m_1}, \mathbf{x}_{m_2}], \\ D^r L_{ij}^{(2k_i:2k_j)} = 0 \text{ for } r = \{1, 2, \cdots 2l\}, \\ \text{and } D^{2l+1} L_{ij}^{(2k_i:2k_j)} < 0. \end{array} \right. \right\}
\tag{2.103}
$$

From the local maximum set of $L_{ij}^{(2k_i:2k_j)}(t_m)$, the global maximum value of $L_{ij}^{(2k_i:2k_j)}(t_m)$ is defined as follows.

Definition 2.29 For a discontinuous dynamical system in Eq. (2.1), there is a point $\mathbf{x}^{(0)}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ at time t_m between two adjacent domains Ω_α $(\alpha = i, j)$. Suppose $\mathbf{x}^{(\alpha)}(t_{m\pm}) = \mathbf{x}_m$ $(\alpha \in \{i,j\})$. For an arbitrarily small $\varepsilon > 0$, there is a time interval ($[t_{m-\varepsilon}, t_m)$ or $(t_m, t_{m+\varepsilon}]$). The flow $\mathbf{x}^{(i)}(t)$ is $C^{r_i}_{[t_{m-\varepsilon},t_m)}$ or $C^{r_i}_{(t_m,t_{m+\varepsilon}]}$-continuous $(r_i \geq 2k_i + 1)$ for time t and $||d^{r_i+1}\mathbf{x}^{(i)}/dt^{r_i+1}|| < \infty$. The flow $\mathbf{x}^{(j)}(t)$ is $C^{r_j}_{[t_{m-\varepsilon},t_m)}$ or $C^{r_j}_{(t_m,t_{m+\varepsilon}]}$-continuous $(r_j \geq 2k_j + 1)$ and $||d^{r_j+1}\mathbf{x}^{(j)}/dt^{r_j+1}|| < \infty$ for time t. The global maximum of *the* $(2k_i : 2k_j)$ *product of G-functions* (i.e., $L_{ij}^{(2k_i:2k_j)}(t_m)$) is defined by

$$
{}_{G\max}L_{ij}^{(2k_i:2k_j)}(t_m) = \max_{t_m \in [t_{m_1}, t_{m_2}]} \left\{ {}_{\max}L_{ij}^{(2k_i:2k_j)}(t_m), L_{ij}^{(2k_i:2k_j)}(t_{m_1}), L_{ij}^{(2k_i:2k_j)}(t_{m_2}) \right\}.
\tag{2.104}
$$

Definition 2.30 For a discontinuous dynamical system in Eq. (2.1), there is a point $\mathbf{x}^{(0)}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ at time t_m between two adjacent domains Ω_α $(\alpha = i, j)$. Suppose $\mathbf{x}^{(i)}(t_{m-}) = \mathbf{x}_m = \mathbf{x}^{(j)}(t_{m\pm})$. For an arbitrarily small $\varepsilon > 0$, there are two time intervals $[t_{m-\varepsilon}, t_m)$ and $(t_m, t_{m+\varepsilon}]$. Both flows $\mathbf{x}^{(i)}(t)$ and $\mathbf{x}^{(j)}(t)$ are $C^{r_i}_{[t_{m-\varepsilon},t_m)}$ and $C^{r_j}_{[t_{m-\varepsilon},t_{m+\varepsilon}]}$-continuous $(r_\alpha \geq 2$ and $\alpha = i, j)$ for time t, respectively, and $||d^{r_\alpha+1}\mathbf{x}^{(\alpha)}/dt^{r_\alpha+1}|| < \infty$. The tangential bifurcation of the flow $\mathbf{x}^{(j)}(t)$ at point $(\mathbf{x}_m, t_m)$ on the boundary $\overrightarrow{\partial\Omega}_{ij}$ is termed *the switching bifurcation of the first kind of the non-passable flow* (or called *the sliding bifurcation*) if

$$
\begin{aligned}
&G_{\partial\Omega_{ij}}^{(j)}(\mathbf{x}_m, t_{m\pm}, \mathbf{p}_j, \boldsymbol{\lambda}) = 0, \\
&G_{\partial\Omega_{ij}}^{(i)}(\mathbf{x}_m, t_{m-}, \mathbf{p}_i, \boldsymbol{\lambda}) \neq 0, \\
&G_{\partial\Omega_{ij}}^{(1,j)}(\mathbf{x}_m, t_{m\pm}, \mathbf{p}_j, \boldsymbol{\lambda}) \neq 0;
\end{aligned}
\tag{2.105}
$$

$$\left.\begin{array}{ll}\text{either} & \left.\begin{array}{l}\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}_{m-\varepsilon})\cdot[\mathbf{x}^{(0)}_{m-\varepsilon}-\mathbf{x}^{(i)}_{m-\varepsilon}]>0;\\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}_{m-\varepsilon})\cdot[\mathbf{x}^{(0)}_{m-\varepsilon}-\mathbf{x}^{(j)}_{m-\varepsilon}]<0,\\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}_{m+\varepsilon})\cdot[\mathbf{x}^{(j)}_{m+\varepsilon}-\mathbf{x}^{(0)}_{m+\varepsilon}]>0\end{array}\right\}\text{for } \mathbf{n}_{\partial\Omega_{ij}}\to\Omega_j\\ \text{or} & \left.\begin{array}{l}\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}_{m-\varepsilon})\cdot[\mathbf{x}^{(0)}_{m-\varepsilon}-\mathbf{x}^{(i)}_{m-\varepsilon}]<0;\\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}_{m-\varepsilon})\cdot[\mathbf{x}^{(0)}_{m-\varepsilon}-\mathbf{x}^{(j)}_{m-\varepsilon}]>0,\\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}_{m+\varepsilon})\cdot[\mathbf{x}^{(j)}_{m+\varepsilon}-\mathbf{x}^{(0)}_{m+\varepsilon}]<0\end{array}\right\}\text{for } \mathbf{n}_{\partial\Omega_{ij}}\to\Omega_i.\end{array}\right\}\quad(2.106)$$

Theorem 2.15 *For a discontinuous dynamical system in Eq. (2.1), there is a point $\mathbf{x}^{(0)}(t_m)\equiv\mathbf{x}_m\in\partial\Omega_{ij}$ at time t_m between two adjacent domains Ω_α $(\alpha=i,j)$. Suppose $\mathbf{x}^{(i)}(t_{m-})=\mathbf{x}_m=\mathbf{x}^{(j)}(t_{m\pm})$. For an arbitrarily small $\varepsilon>0$, there are two time intervals $[t_{m-\varepsilon},t_m)$ and $(t_m,t_{m+\varepsilon}]$. The flows $\mathbf{x}^{(i)}(t)$ and $\mathbf{x}^{(j)}(t)$ are $C^{r_i}_{[t_{m-\varepsilon},t_m)}$ and $C^{r_j}_{[t_{m-\varepsilon},t_{m+\varepsilon}]}$-continuous for time t and $\|d^{r_\alpha+1}\mathbf{x}^{(\alpha)}/dt^{r_\alpha+1}\|<\infty$ $(r_\alpha\geq 3$ and $\alpha=i,j)$. The sliding bifurcation of the passable flow of $\mathbf{x}^{(i)}(t)$ and $\mathbf{x}^{(j)}(t)$ at point $(\mathbf{x}_m,t_m)$ switching to the non-passable flow of the first kind on the boundary $\overrightarrow{\partial\Omega}_{ij}$ occurs if and only if*

$$\left.\begin{array}{r}\left.\begin{array}{l}G^{(j)}_{\partial\Omega_{ij}}(\mathbf{x}_m,t_{m\pm},\mathbf{p}_j,\boldsymbol{\lambda})=0 \text{ or}\\ L_{ij}(\mathbf{x}_m,t_m,\mathbf{p}_i,\mathbf{p}_j,\boldsymbol{\lambda})=0 \text{ or}\\ {}_{G\,\min}L_{ij}(t_m)=0;\end{array}\right\}\\ \left.\begin{array}{l}G^{(i)}_{\partial\Omega_{ij}}(\mathbf{x}_m,t_{m-},\mathbf{p}_i,\boldsymbol{\lambda})>0 \text{ for } \mathbf{n}_{\partial\Omega_{ij}}\to\Omega_j\\ G^{(i)}_{\partial\Omega_{ij}}(\mathbf{x}_m,t_{m-},\mathbf{p}_i,\boldsymbol{\lambda})<0 \text{ for } \mathbf{n}_{\partial\Omega_{ij}}\to\Omega_i,\end{array}\right\}\end{array}\right.\quad(2.107)$$

$$\left.\begin{array}{l}G^{(1,j)}_{\partial\Omega_{ij}}(\mathbf{x}_m,t_{m\pm},\mathbf{p}_j,\boldsymbol{\lambda})>0 \text{ for } \mathbf{n}_{\partial\Omega_{ij}}\to\Omega_j\\ G^{(1,j)}_{\partial\Omega_{ij}}(\mathbf{x}_m,t_{m\pm},\mathbf{p}_j,\boldsymbol{\lambda})<0 \text{ for } \mathbf{n}_{\partial\Omega_{ij}}\to\Omega_i.\end{array}\right\}\quad(2.108)$$

Proof The proof is the same as in the proof of Theorem 2.1 and Theorem 2.2. This theorem can be proved. □

Definition 2.31 For a discontinuous dynamical system in Eq. (2.1), there is a point $\mathbf{x}^{(0)}(t_m)\equiv\mathbf{x}_m\in\partial\Omega_{ij}$ at time t_m between two adjacent domains Ω_α $(\alpha=i,j)$. Suppose $\mathbf{x}^{(i)}(t_{m-})=\mathbf{x}_m=\mathbf{x}^{(j)}(t_{m\pm})$. For an arbitrarily small $\varepsilon>0$, there are two time intervals (i.e., $[t_{m-\varepsilon},t_m)$ and $(t_m,t_{m+\varepsilon}]$). A flow $\mathbf{x}^{(i)}(t)$ is $C^{r_i}_{[t_{m-\varepsilon},t_m)}$-continuous

$(r_i \geq 2k_i + 1)$ and $\|d^{r_i+1}\mathbf{x}^{(i)}/dt^{r_i+1}\| < \infty$ for time t, and a flow $\mathbf{x}^{(j)}(t)$ is $C^{r_j}_{[t_{m-\varepsilon}, t_{m+\varepsilon}]}$-continuous $(r_j \geq 2k_j + 1)$ and $\|d^{r_j+1}\mathbf{x}^{(j)}/dt^{r_j+1}\| < \infty$. The bifurcation of the $(2k_i : 2k_j)$-passable flow of $\mathbf{x}^{(i)}(t)$ and $\mathbf{x}^{(j)}(t)$ at point $(\mathbf{x}_m, t_m)$ on the boundary $\overrightarrow{\partial\Omega}_{ij}$ is termed *the switching bifurcation of the first kind of the* $(2k_i : 2k_j)$*-non-passable flow* (or called *the* $(2k_i : 2k_j)$*-sliding bifurcation*) if

$$\begin{aligned}
&G^{(s_j,j)}_{\partial\Omega_{ij}}(\mathbf{x}_m, t_{m\pm}, \mathbf{p}_j, \boldsymbol{\lambda}) = 0 \text{ for } s_j = 0, 1, \cdots, 2k_j,\\
&G^{(s_i,i)}_{\partial\Omega_{ij}}(\mathbf{x}_m, t_{m-}, \mathbf{p}_i, \boldsymbol{\lambda}) = 0 \text{ for } s_i = 0, 1, \cdots, 2k_i - 1,\\
&G^{(2k_i,i)}_{\partial\Omega_{ij}}(\mathbf{x}_m, t_{m-}, \mathbf{p}_i, \boldsymbol{\lambda}) \neq 0 \text{ and } G^{(2k_j+1,\,j)}_{\partial\Omega_{ij}}(\mathbf{x}_m, t_{m\pm}, \mathbf{p}_j, \boldsymbol{\lambda}) \neq 0;
\end{aligned} \tag{2.109}$$

$$\begin{aligned}
&\text{either} \quad \left.\begin{aligned}
&\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}_{m-\varepsilon}) \cdot [\mathbf{x}^{(0)}_{m-\varepsilon} - \mathbf{x}^{(j)}_{m-\varepsilon}] < 0,\\
&\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}_{m+\varepsilon}) \cdot [\mathbf{x}^{(j)}_{m+\varepsilon} - \mathbf{x}^{(0)}_{m+\varepsilon}] > 0;\\
&\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}_{m-\varepsilon}) \cdot [\mathbf{x}^{(0)}_{m-\varepsilon} - \mathbf{x}^{(i)}_{m-\varepsilon}] > 0
\end{aligned}\right\} \text{for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_j,\\
&\text{or} \quad \left.\begin{aligned}
&\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}_{m-\varepsilon}) \cdot [\mathbf{x}^{(0)}_{m-\varepsilon} - \mathbf{x}^{(j)}_{m-\varepsilon}] > 0,\\
&\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}_{m+\varepsilon}) \cdot [\mathbf{x}^{(j)}_{m+\varepsilon} - \mathbf{x}^{(0)}_{m+\varepsilon}] < 0;\\
&\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}_{m-\varepsilon}) \cdot [\mathbf{x}^{(0)}_{m-\varepsilon} - \mathbf{x}^{(i)}_{m-\varepsilon}] < 0
\end{aligned}\right\} \text{for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_i.
\end{aligned} \tag{2.110}$$

Theorem 2.16 *For a discontinuous dynamical system in Eq. (2.1), there is a point* $\mathbf{x}^{(0)}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ *at time* t_m *between two adjacent domains* Ω_α $(\alpha = i, j)$*. Suppose* $\mathbf{x}^{(i)}(t_{m-}) = \mathbf{x}_m = \mathbf{x}^{(j)}(t_{m\pm})$*. For an arbitrarily small* $\varepsilon > 0$*, there are two time intervals* $[t_{m-\varepsilon}, t_m)$ *and* $(t_m, t_{m+\varepsilon}]$*. A flow* $\mathbf{x}^{(i)}(t)$ *is* $C^{r_i}_{[t_{m-\varepsilon}, t_m)}$*-continuous* $(r_i \geq 2k_i + 1)$ *and* $\|d^{r_i+1}\mathbf{x}^{(i)}/dt^{r_i+1}\| < \infty$ *for time t, and a flow* $\mathbf{x}^{(j)}(t)$ *is* $C^{r_j}_{[t_{m-\varepsilon}, t_{m+\varepsilon}]}$*-continuous* $(r_j \geq 2k_j + 1)$ *and* $\|d^{r_j+1}\mathbf{x}^{(j)}/dt^{r_j+1}\| < \infty$*. The sliding bifurcation of the* $(2k_i : 2k_j)$*-passable flow of* $\mathbf{x}^{(i)}(t)$ *and* $\mathbf{x}^{(j)}(t)$ *at point* $(\mathbf{x}_m, t_m)$ *switching to the* $(2k_i : 2k_j)$*-non-passable flow of the first kind on the boundary* $\overrightarrow{\partial\Omega}_{ij}$ *(a* $(2k_i : 2k_j)$*-sliding bifurcation) occurs if and only if*

$$\left.\begin{aligned}
&G^{(s_j,j)}_{\partial\Omega_{ij}}(\mathbf{x}_m, t_{m\pm}, \mathbf{p}_j, \boldsymbol{\lambda}) = 0 \text{ for } s_j = 0, 1, \cdots, 2k_j - 1;\\
&G^{(s_i,i)}_{\partial\Omega_{ij}}(\mathbf{x}_m, t_{m-}, \mathbf{p}_i, \boldsymbol{\lambda}) = 0 \text{ for } s_i = 0, 1, \cdots, 2k_i - 1;
\end{aligned}\right\} \tag{2.111}$$

$$\left.\begin{aligned}
&G^{(2k_j,j)}_{\partial\Omega_{ij}}(\mathbf{x}_m, t_{m\pm}, \mathbf{p}_j, \boldsymbol{\lambda}) = 0, \text{ or}\\
&L^{(2k_i:2k_j)}_{ij}(\mathbf{x}_m, t_m, \mathbf{p}_i, \mathbf{p}_j, \boldsymbol{\lambda}) = 0, \text{or}\\
&{}_{G\,\min}L^{(2k_i:2k_j)}_{ij}(t_m) = 0,
\end{aligned}\right\} \tag{2.112}$$

$$\left.\begin{array}{l} G^{(2k_i,i)}_{\partial\Omega_{ij}}(\mathbf{x}_m, t_{m-}, \mathbf{p}_i, \boldsymbol{\lambda}) > 0 \text{ for } \mathbf{n}_{\partial\Omega_{ij}} \rightarrow \Omega_j, \\ G^{(2k_i,i)}_{\partial\Omega_{ij}}(\mathbf{x}_m, t_{m-}, \mathbf{p}_i, \boldsymbol{\lambda}) < 0 \text{ for } \mathbf{n}_{\partial\Omega_{ij}} \rightarrow \Omega_i; \end{array}\right\}$$

$$\left.\begin{array}{l} G^{(2k_j+1,j)}_{\partial\Omega_{ij}}(\mathbf{x}_m, t_{m\pm}, \mathbf{p}_j, \boldsymbol{\lambda}) > 0 \text{ for } \mathbf{n}_{\partial\Omega_{ij}} \rightarrow \Omega_j, \\ G^{(2k_j+1,j)}_{\partial\Omega_{ij}}(\mathbf{x}_m, t_{m\pm}, \mathbf{p}_j, \boldsymbol{\lambda}) < 0 \text{ for } \mathbf{n}_{\partial\Omega_{ij}} \rightarrow \Omega_i. \end{array}\right\} \quad (2.113)$$

Proof The proof is the same as in the proof of Theorem 2.1 and Theorem 2.2. This theorem can be proved. □

Definition 2.32 For a discontinuous dynamical system in Eq. (2.1), there is a point $\mathbf{x}^{(0)}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ at time t_m between two adjacent domains Ω_α ($\alpha = i, j$). Suppose $\mathbf{x}^{(i)}(t_{m\pm}) = \mathbf{x}_m = \mathbf{x}^{(j)}(t_{m+})$. For an arbitrarily small $\varepsilon > 0$, there are two time intervals $[t_{m-\varepsilon}, t_m)$ and $(t_m, t_{m+\varepsilon}]$. Both flows $\mathbf{x}^{(i)}(t)$ and $\mathbf{x}^{(j)}(t)$ are $C^{r_i}_{[t_{m-\varepsilon}, t_{m+\varepsilon}]}$ and $C^{r_j}_{[t_{m-\varepsilon}, t_m)}$-continuous ($r_\alpha \geq 2$ and $\alpha = i, j$) for time t, respectively, and $||d^{r_\alpha+1}\mathbf{x}^{(\alpha)}/dt^{r_\alpha+1}|| < \infty$. The tangential bifurcation of the flow $\mathbf{x}^{(i)}(t)$ at point $(\mathbf{x}_m, t_m)$ on the boundary $\overrightarrow{\partial\Omega}_{ij}$ is termed *a switching bifurcation of the non-passable flow of the second kind* (or called *a source bifurcation*) if

$$\left.\begin{array}{l} G^{(j)}_{\partial\Omega_{ij}}(\mathbf{x}_m, t_{m+}, \mathbf{p}_j, \boldsymbol{\lambda}) \neq 0, \\ G^{(i)}_{\partial\Omega_{ij}}(\mathbf{x}_m, t_{m\mp}, \mathbf{p}_i, \boldsymbol{\lambda}) = 0, \\ G^{(1,i)}_{\partial\Omega_{ij}}(\mathbf{x}_m, t_{m\mp}, \mathbf{p}_i, \boldsymbol{\lambda}) \neq 0; \end{array}\right\} \quad (2.114)$$

$$\text{either} \quad \left.\begin{array}{l} \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}_{m-\varepsilon}) \cdot [\mathbf{x}^{(0)}_{m-\varepsilon} - \mathbf{x}^{(i)}_{m-\varepsilon}] > 0, \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}_{m+\varepsilon}) \cdot [\mathbf{x}^{(i)}_{m+\varepsilon} - \mathbf{x}^{(0)}_{m+\varepsilon}] < 0, \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}_{m+\varepsilon}) \cdot [\mathbf{x}^{(j)}_{m+\varepsilon} - \mathbf{x}^{(0)}_{m+\varepsilon}] > 0 \end{array}\right\} \text{for } \mathbf{n}_{\partial\Omega_{ij}} \rightarrow \Omega_j$$

$$\text{or} \quad \left.\begin{array}{l} \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}_{m-\varepsilon}) \cdot [\mathbf{x}^{(0)}_{m-\varepsilon} - \mathbf{x}^{(i)}_{m-\varepsilon}] < 0, \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}_{m+\varepsilon}) \cdot [\mathbf{x}^{(i)}_{m+\varepsilon} - \mathbf{x}^{(0)}_{m+\varepsilon}] > 0, \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}_{m+\varepsilon}) \cdot [\mathbf{x}^{(j)}_{m+\varepsilon} - \mathbf{x}^{(0)}_{m+\varepsilon}] < 0 \end{array}\right\} \text{for } \mathbf{n}_{\partial\Omega_{ij}} \rightarrow \Omega_i. \quad (2.115)$$

Theorem 2.17 *For a discontinuous dynamical system in Eq. (2.1), there is a point $\mathbf{x}^{(0)}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ at time t_m between two adjacent domains Ω_α ($\alpha = i, j$). Suppose $\mathbf{x}^{(i)}(t_{m-}) = \mathbf{x}_m = \mathbf{x}^{(j)}(t_{m+})$. For an arbitrarily small $\varepsilon > 0$, there are two time intervals $[t_{m-\varepsilon}, t_m)$ and $(t_m, t_{m+\varepsilon}]$. Both flows $\mathbf{x}^{(i)}(t)$ and $\mathbf{x}^{(j)}(t)$ are $C^{r_i}_{[t_{m-\varepsilon}, t_{m+\varepsilon}]}$ and*

$C^{r_j}_{[t_{m-\varepsilon},t_m)}$*-continuous for time t with* $||d^{r_\alpha+1}\mathbf{x}^{(\alpha)}/dt^{r_\alpha+1}|| < \infty$ $(r_\alpha \geq 2,\ \alpha = i,j)$. *The source bifurcation of the passable flow of* $\mathbf{x}^{(i)}(t)$ *and* $\mathbf{x}^{(j)}(t)$ *at point* $(\mathbf{x}_m, t_m)$ *switching to the non-passable flow of the second kind on the boundary* $\overrightarrow{\partial\Omega}_{ij}$ *occurs if and only if*

$$\left.\begin{aligned} &G^{(i)}_{\partial\Omega_{ij}}(\mathbf{x}_m, t_{m\mp}, \mathbf{p}_i, \boldsymbol{\lambda}) = 0, \text{ or} \\ &L_{ij}(\mathbf{x}_m, t_m, \mathbf{p}_i, \mathbf{p}_j, \boldsymbol{\lambda}) = 0, \text{ or} \\ &{}_{G\,\min}L_{ij}(t_m) = 0, \end{aligned}\right\}$$
$$\left.\begin{aligned} &G^{(j)}_{\partial\Omega_{ij}}(\mathbf{x}_m, t_{m+}, \mathbf{p}_j, \boldsymbol{\lambda}) > 0 \text{ for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_j, \\ &G^{(j)}_{\partial\Omega_{ij}}(\mathbf{x}_m, t_{m+}, \mathbf{p}_j, \boldsymbol{\lambda}) < 0 \text{ for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_i; \end{aligned}\right\} \tag{2.116}$$

$$\left.\begin{aligned} &G^{(1,i)}_{\partial\Omega_{ij}}(\mathbf{x}_m, t_{m\mp}, \mathbf{p}_i, \boldsymbol{\lambda}) < 0 \text{ for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_j \\ &G^{(1,i)}_{\partial\Omega_{ij}}(\mathbf{x}_m, t_{m\mp}, \mathbf{p}_i, \boldsymbol{\lambda}) > 0 \text{ for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_i. \end{aligned}\right\} \tag{2.117}$$

Proof The proof is the same as in the proof of Theorem 2.1 and Theorem 2.2. This theorem can be proved. □

Definition 2.33 For a discontinuous dynamical system in Eq. (2.1), there is a point $\mathbf{x}^{(0)}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ at time t_m between two adjacent domains Ω_α $(\alpha = i,j)$. Suppose $\mathbf{x}^{(i)}(t_{m\pm}) = \mathbf{x}_m = \mathbf{x}^{(j)}(t_{m+})$. For an arbitrarily small $\varepsilon > 0$, there are two time intervals $[t_{m-\varepsilon}, t_m)$ and $(t_m, t_{m+\varepsilon}]$. A flow $\mathbf{x}^{(i)}(t)$ is $C^{r_i}_{[t_{m-\varepsilon},t_{m+\varepsilon}]}$-continuous $(r_i \geq 2k_i + 2)$ with $||d^{r_i+1}\mathbf{x}^{(i)}/dt^{r_i+1}|| < \infty$ for time t, and a flow $\mathbf{x}^{(j)}(t)$ is $C^{r_j}_{(t_m,t_{m+\varepsilon}]}$-continuous for time t and $||d^{r_j+1}\mathbf{x}^{(j)}/dt^{r_j+1}|| < \infty$ $(r_j \geq 2k_j + 1)$. The tangential bifurcation of the $(2k_i : 2k_j)$-passable flow of $\mathbf{x}^{(i)}(t)$ and $\mathbf{x}^{(j)}(t)$ at point $(\mathbf{x}_m, t_m)$ on the boundary $\overrightarrow{\partial\Omega}_{ij}$ is termed *a switching bifurcation of the* $(2k_i : 2k_j)$, *non-passable flow of the second kind* (or called *a* $(2k_i : 2k_j)$*-source bifurcation*) if

$$\left.\begin{aligned} &G^{(r_i,i)}_{\partial\Omega_{ij}}(\mathbf{x}_m, t_{m\mp}, \mathbf{p}_i, \boldsymbol{\lambda}) = 0 \text{ for } r_i = 0, 1, \cdots, 2k_i; \\ &G^{(r_j,j)}_{\partial\Omega_{ij}}(\mathbf{x}_m, t_{m+}, \mathbf{p}_j, \boldsymbol{\lambda}) = 0 \text{ for } r_j = 0, 1, \cdots, 2k_j - 1; \\ &G^{(2k_i+1,i)}_{\partial\Omega_{ij}}(\mathbf{x}_m, t_{m\mp}, \mathbf{p}_i, \boldsymbol{\lambda}) \neq 0; \\ &G^{(2k_j,j)}_{\partial\Omega_{ij}}(\mathbf{x}_m, t_{m+}, \mathbf{p}_j, \boldsymbol{\lambda}) \neq 0; \end{aligned}\right\} \tag{2.118}$$

$$\text{either} \quad \left.\begin{aligned} &\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}_{m-\varepsilon}) \cdot [\mathbf{x}^{(0)}_{m-\varepsilon} - \mathbf{x}^{(i)}_{m-\varepsilon}] > 0 \\ &\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}_{m+\varepsilon}) \cdot [\mathbf{x}^{(i)}_{m+\varepsilon} - \mathbf{x}^{(0)}_{m+\varepsilon}] < 0 \\ &\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}_{m+\varepsilon}) \cdot [\mathbf{x}^{(j)}_{m+\varepsilon} - \mathbf{x}^{(0)}_{m+\varepsilon}] > 0 \end{aligned}\right\} \text{for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_j,$$

$$\text{or} \quad \left.\begin{array}{l} \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}_{m-\varepsilon}) \cdot [\mathbf{x}^{(0)}_{m-\varepsilon} - \mathbf{x}^{(i)}_{m-\varepsilon}] < 0 \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}_{m+\varepsilon}) \cdot [\mathbf{x}^{(i)}_{m+\varepsilon} - \mathbf{x}^{(0)}_{m+\varepsilon}] > 0 \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}_{m+\varepsilon}) \cdot [\mathbf{x}^{(j)}_{m+\varepsilon} - \mathbf{x}^{(0)}_{m+\varepsilon}] < 0 \end{array}\right\} \text{for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_i. \tag{2.119}$$

Theorem 2.18 *For a discontinuous dynamical system in Eq. (2.1), there is a point $\mathbf{x}^{(0)}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ at time t_m between two adjacent domains Ω_α ($\alpha = i, j$). Suppose $\mathbf{x}^{(i)}(t_{m\mp}) = \mathbf{x}_m = \mathbf{x}^{(j)}(t_{m+})$. For an arbitrarily small $\varepsilon > 0$, there are two time intervals $[t_{m-\varepsilon}, t_m)$ and $(t_m, t_{m+\varepsilon}]$. A flow $\mathbf{x}^{(i)}(t)$ is $C^{r_i}_{[t_{m-\varepsilon}, t_{m+\varepsilon}]}$-continuous ($r_i \geq 2k_i + 2$) with $||d^{r_i+1}\mathbf{x}^{(i)}/dt^{r_i+1}|| < \infty$ for time t, and a flow $\mathbf{x}^{(j)}(t)$ is $C^{r_j}_{(t_m, t_{m+\varepsilon}]}$-continuous ($r_j \geq 2k_j + 1$) with $||d^{r_j+1}\mathbf{x}^{(j)}/dt^{r_j+1}|| < \infty$. The source bifurcation of the $(2k_i : 2k_j)$-passable flow of $\mathbf{x}^{(i)}(t)$ and $\mathbf{x}^{(j)}(t)$ at point $(\mathbf{x}_m, t_m)$ switching to the $(2k_i : 2k_j)$-non-passable flow of the second kind on the boundary $\overrightarrow{\partial\Omega}_{ij}$ (or the $(2k_i : 2k_j)$-source bifurcation) occurs if and only if*

$$\left.\begin{array}{l} G^{(r_i,i)}_{\partial\Omega_{ij}}(\mathbf{x}_m, t_{m\mp}, \mathbf{p}_i, \boldsymbol{\lambda}) = 0 \text{ for } r_i = 0, 1, \cdots, 2k_i - 1; \\ G^{(r_j,j)}_{\partial\Omega_{ij}}(\mathbf{x}_m, t_{m+}, \mathbf{p}_j, \boldsymbol{\lambda}) = 0 \text{ for } r_j = 0, 1, \cdots, 2k_j - 1; \end{array}\right\} \tag{2.120}$$

$$\left.\begin{array}{l} \left.\begin{array}{l} G^{(2k_i,i)}_{\partial\Omega_{ij}}(\mathbf{x}_m, t_{m\mp}, \mathbf{p}_i, \boldsymbol{\lambda}) = 0, \text{ or} \\ L^{(2k_i:2k_j)}_{ij}(\mathbf{x}_m, t_m, \mathbf{p}_i, \mathbf{p}_j, \boldsymbol{\lambda}) = 0, \text{ or} \\ {}_{G\,\min}L^{(2k_i:2k_j)}_{ij}(t_m) = 0; \end{array}\right\} \\ \left.\begin{array}{l} G^{(2k_j,j)}_{\partial\Omega_{ij}}(\mathbf{x}_m, t_{m+}, \mathbf{p}_j, \boldsymbol{\lambda}) > 0 \text{ for } \mathbf{n}_{\Omega_{ij}} \to \Omega_j, \\ G^{(2k_j,j)}_{\partial\Omega_{ij}}(\mathbf{x}_m, t_{m+}, \mathbf{p}_j, \boldsymbol{\lambda}) < 0 \text{ for } \mathbf{n}_{\Omega_{ij}} \to \Omega_i; \end{array}\right\} \end{array}\right. \tag{2.121}$$

$$\left.\begin{array}{l} G^{(2k_i+1,i)}_{\partial\Omega_{ij}}(\mathbf{x}_m, t_{m\mp}, \mathbf{p}_i, \boldsymbol{\lambda}) < 0 \text{ for } \mathbf{n}_{\Omega_{ij}} \to \Omega_j \\ G^{(2k_i+1,i)}_{\partial\Omega_{ij}}(\mathbf{x}_m, t_{m\mp}, \mathbf{p}_i, \boldsymbol{\lambda}) > 0 \text{ for } \mathbf{n}_{\Omega_{ij}} \to \Omega_i. \end{array}\right\} \tag{2.122}$$

Proof The proof is the same as in the proof of Theorem 2.1 and Theorem 2.2. This theorem can be proved. □

Definition 2.34 For a discontinuous dynamical system in Eq. (2.1), there is a point $\mathbf{x}^{(0)}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ at time t_m between two adjacent domains Ω_α ($\alpha = i, j$). Suppose $\mathbf{x}^{(i)}(t_{m-}) = \mathbf{x}_m = \mathbf{x}^{(j)}(t_{m+})$. For an arbitrarily small $\varepsilon > 0$, there are two time intervals $[t_{m-\varepsilon}, t_m)$ and $(t_m, t_{m+\varepsilon}]$. The flow $\mathbf{x}^{(\alpha)}(t)$ is $C^{r_\alpha}_{[t_{m-\varepsilon}, t_{m+\varepsilon}]}$-continuous for time t, and $||d^{r_\alpha+1}\mathbf{x}^{(\alpha)}/dt^{r_\alpha+1}|| < \infty$ ($r_\alpha \geq 2$, $\alpha = i, j$). The tangential bifurcations of two flows $\mathbf{x}^{(i)}(t)$ and $\mathbf{x}^{(j)}(t)$ at point $(\mathbf{x}_m, t_m)$ on the boundary $\overrightarrow{\partial\Omega}_{ij}$ are termed *a switching bifurcation of the flow from* $\overrightarrow{\partial\Omega}_{ij}$ to $\overleftarrow{\partial\Omega}_{ij}$ if

$$\left.\begin{array}{l} G^{(i)}_{\partial\Omega_{ij}}(\mathbf{x}_m, t_{m\mp}, \mathbf{p}_i, \boldsymbol{\lambda}) = 0 \text{ and } G^{(j)}_{\partial\Omega_{ij}}(\mathbf{x}_m, t_{m\pm}, \mathbf{p}_j, \boldsymbol{\lambda}) = 0, \\ G^{(1,i)}_{\partial\Omega_{ij}}(\mathbf{x}_m, t_{m\mp}, \mathbf{p}_i, \boldsymbol{\lambda}) \neq 0 \text{ and } G^{(1,j)}_{\partial\Omega_{ij}}(\mathbf{x}_m, t_{m\pm}, \mathbf{p}_j, \boldsymbol{\lambda}) \neq 0; \end{array}\right\} \tag{2.123}$$

$$\left.\begin{array}{l} \text{either} \left.\begin{array}{l} \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}_{m-\varepsilon}) \cdot [\mathbf{x}^{(0)}_{m-\varepsilon} - \mathbf{x}^{(i)}_{m-\varepsilon}] > 0, \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}_{m+\varepsilon}) \cdot [\mathbf{x}^{(i)}_{m+\varepsilon} - \mathbf{x}^{(0)}_{m+\varepsilon}] < 0; \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}_{m-\varepsilon}) \cdot [\mathbf{x}^{(0)}_{m-\varepsilon} - \mathbf{x}^{(j)}_{m-\varepsilon}] < 0, \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}_{m+\varepsilon}) \cdot [\mathbf{x}^{(j)}_{m+\varepsilon} - \mathbf{x}^{(0)}_{m+\varepsilon}] > 0 \end{array}\right\} \text{for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_j \\ \text{or} \left.\begin{array}{l} \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}_{m-\varepsilon}) \cdot [\mathbf{x}^{(0)}_{m-\varepsilon} - \mathbf{x}^{(i)}_{m-\varepsilon}] < 0, \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}_{m+\varepsilon}) \cdot [\mathbf{x}^{(i)}_{m+\varepsilon} - \mathbf{x}^{(0)}_{m+\varepsilon}] > 0; \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}_{m-\varepsilon}) \cdot [\mathbf{x}^{(0)}_{m-\varepsilon} - \mathbf{x}^{(j)}_{m-\varepsilon}] > 0, \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}_{m+\varepsilon}) \cdot [\mathbf{x}^{(j)}_{m+\varepsilon} - \mathbf{x}^{(0)}_{m+\varepsilon}] < 0 \end{array}\right\} \text{for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_i. \end{array}\right\} \tag{2.124}$$

Theorem 2.19 *For a discontinuous dynamical system in Eq. (2.1), there is a point* $\mathbf{x}^{(0)}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ *at time* t_m *between two adjacent domains* Ω_α $(\alpha = i, j)$. *Suppose* $\mathbf{x}^{(i)}(t_{m\pm}) = \mathbf{x}_m = \mathbf{x}^{(j)}(t_{m\pm})$. *For an arbitrarily small* $\varepsilon > 0$, *there is a time interval* $[t_{m-\varepsilon}, t_{m+\varepsilon}]$. *The flow* $\mathbf{x}^{(\alpha)}(t)$ *is* $C^{r_\alpha}_{[t_{m-\varepsilon}, t_{m+\varepsilon}]}$*-continuous for time t with* $\|d^{r_\alpha+1}\mathbf{x}^{(\alpha)}/dt^{r_\alpha+1}\| < \infty$ $(r_\alpha \geq 3, \alpha = i, j)$. *The tangential bifurcations of the flow* $\mathbf{x}^{(i)}(t)$ *and* $\mathbf{x}^{(j)}(t)$ *at point* $(\mathbf{x}_m, t_m)$ *on the boundary* $\overrightarrow{\partial\Omega}_{ij}$ (or *the switching bifurcation of the flow from* $\overrightarrow{\partial\Omega}_{ij}$ to $\overleftarrow{\partial\Omega}_{ij}$) *occur if and only if*

$$\left.\begin{array}{l} G^{(i)}_{\partial\Omega_{ij}}(\mathbf{x}_m, t_{m\mp}, \mathbf{p}_j, \boldsymbol{\lambda}) = 0 \text{ and } G^{(j)}_{\partial\Omega_{ij}}(\mathbf{x}_m, t_{m\pm}, \mathbf{p}_i, \boldsymbol{\lambda}) = 0, \text{or} \\ L_{ij}(\mathbf{x}_{m_2}, t_{m_2}, \mathbf{p}_i, \mathbf{p}_j, \boldsymbol{\lambda}) = 0, \text{or} \\ {}_{G\,\min}L_{ij}(t_m) = 0; \end{array}\right\} \tag{2.125}$$

$$\left.\begin{array}{l} \left.\begin{array}{l} G^{(1,i)}_{\partial\Omega_{ij}}(\mathbf{x}_m, t_{m\mp}, \mathbf{p}_i, \boldsymbol{\lambda}) < 0, \\ G^{(1,j)}_{\partial\Omega_{ij}}(\mathbf{x}_m, t_{m\pm}, \mathbf{p}_j, \boldsymbol{\lambda}) > 0 \end{array}\right\} \text{for } \mathbf{n}_{\Omega_{ij}} \to \Omega_j \\ \left.\begin{array}{l} G^{(1,i)}_{\partial\Omega_{ij}}(\mathbf{x}_m, t_{m\mp}, \mathbf{p}_i, \boldsymbol{\lambda}) > 0, \\ G^{(1,j)}_{\partial\Omega_{ij}}(\mathbf{x}_m, t_{m\pm}, \mathbf{p}_j, \boldsymbol{\lambda}) < 0 \end{array}\right\} \text{for } \mathbf{n}_{\Omega_{ij}} \to \Omega_i. \end{array}\right\} \tag{2.126}$$

Proof The proof is the same as in the proof of Theorem 2.1 and Theorem 2.2. This theorem can be proved. □

Definition 2.35 For a discontinuous dynamical system in Eq. (2.1), there is a point $\mathbf{x}^{(0)}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ at time t_m between two adjacent domains Ω_α $(\alpha = i, j)$. *Suppose* $\mathbf{x}^{(i)}(t_{m\pm}) = \mathbf{x}_m = \mathbf{x}^{(j)}(t_{m\pm})$. For an arbitrarily small $\varepsilon > 0$, there is a time interval $[t_{m-\varepsilon}, t_{m+\varepsilon}]$. A flow $\mathbf{x}^{(\alpha)}(t)$ is $C^{r_\alpha}_{[t_{m-\varepsilon}, t_{m+\varepsilon}]}$-continuous for time t with $||d^{r_\alpha+1}\mathbf{x}^{(\alpha)}/dt^{r_\alpha+1}|| < \infty$ $(r_\alpha \geq 2k_\alpha + 1,\ \alpha = i, j)$. The tangential bifurcation of the $(2k_i : 2k_j)$-passable flow of $\mathbf{x}^{(i)}(t)$ and $\mathbf{x}^{(j)}(t)$ at point $(\mathbf{x}_m, t_m)$ on the boundary $\overrightarrow{\partial\Omega}_{ij}$ is termed *a switching bifurcation of the* $(2k_j : 2k_i)$*-passable flow from* $\overrightarrow{\partial\Omega}_{ij}$ to $\overleftarrow{\partial\Omega}_{ij}$ if

$$\left.\begin{array}{l} G^{(s,i)}_{\partial\Omega_{ij}}(\mathbf{x}_m, t_{m\mp}, \mathbf{p}_i, \boldsymbol{\lambda}) = 0 \text{ for } s = 0, 1, \cdots, 2k_i \\ G^{(s,j)}_{\partial\Omega_{ij}}(\mathbf{x}_m, t_{m\pm}, \mathbf{p}_j, \boldsymbol{\lambda}) = 0 \text{ for } s = 0, 1, \cdots, 2k_j \\ G^{(2k_i+1,i)}_{\partial\Omega_{ij}}(\mathbf{x}_m, t_{m\mp}, \mathbf{p}_i, \boldsymbol{\lambda}) \neq 0, \\ G^{(2k_j+1,j)}_{\partial\Omega_{ij}}(\mathbf{x}_m, t_{m\pm}, \mathbf{p}_j, \boldsymbol{\lambda}) \neq 0 \end{array}\right\} \tag{2.127}$$

$$\begin{array}{l} \text{either} \left.\begin{array}{l} \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}_{m-\varepsilon}) \cdot [\mathbf{x}^{(0)}_{m-\varepsilon} - \mathbf{x}^{(i)}_{m-\varepsilon}] > 0, \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}_{m+\varepsilon}) \cdot [\mathbf{x}^{(i)}_{m+\varepsilon} - \mathbf{x}^{(0)}_{m+\varepsilon}] < 0; \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}_{m-\varepsilon}) \cdot [\mathbf{x}^{(0)}_{m-\varepsilon} - \mathbf{x}^{(j)}_{m-\varepsilon}] > 0, \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}_{m+\varepsilon}) \cdot [\mathbf{x}^{(j)}_{m+\varepsilon} - \mathbf{x}^{(0)}_{m+\varepsilon}] > 0 \end{array}\right\} \text{for } \mathbf{n}_{\partial\Omega_{ij}} \rightarrow \Omega_j \\ \text{or} \left.\begin{array}{l} \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}_{m-\varepsilon}) \cdot [\mathbf{x}^{(0)}_{m-\varepsilon} - \mathbf{x}^{(i)}_{m-\varepsilon}] < 0, \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}_{m+\varepsilon}) \cdot [\mathbf{x}^{(i)}_{m+\varepsilon} - \mathbf{x}^{(0)}_{m+\varepsilon}] > 0; \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}_{m-\varepsilon}) \cdot [\mathbf{x}^{(0)}_{m-\varepsilon} - \mathbf{x}^{(j)}_{m-\varepsilon}] < 0, \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}_{m+\varepsilon}) \cdot [\mathbf{x}^{(j)}_{m+\varepsilon} - \mathbf{x}^{(0)}_{m+\varepsilon}] < 0 \end{array}\right\} \text{for } \mathbf{n}_{\partial\Omega_{ij}} \rightarrow \Omega_i. \end{array} \tag{2.128}$$

Theorem 2.20 *For a discontinuous dynamical system in Eq. (2.1), there is a point* $\mathbf{x}^{(0)}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ *at time* t_m *between two adjacent domains* Ω_α $(\alpha = i, j)$. *Suppose* $\mathbf{x}^{(i)}(t_{m-}) = \mathbf{x}_m = \mathbf{x}^{(j)}(t_{m+})$. *For an arbitrarily small* $\varepsilon > 0$, *there are two time intervals* $[t_{m-\varepsilon}, t_m)$ *and* $(t_m, t_{m+\varepsilon}]$. *A flow* $\mathbf{x}^{(i)}(t)$ *is* $C^{r_i}_{[t_{m-\varepsilon}, t_m)}$*-continuous* $(r_i \geq 2k_i + 1)$ *with* $||d^{r_i+1}\mathbf{x}^{(i)}/dt^{r_i+1}|| < \infty$ *for time t, and a flow* $\mathbf{x}^{(j)}(t)$ *is* $C^{r_j}_{(t_m, t_{m+\varepsilon}]}$*-continuous* $(r_j \geq 2k_j + 2)$ *with* $||d^{r_j+1}\mathbf{x}^{(j)}/dt^{r_j+1}|| < \infty$. *The bifurcation of the* $(2k_i : 2k_j)$*-passable flow of* $\mathbf{x}^{(i)}(t)$ *and* $\mathbf{x}^{(j)}(t)$ *at point* $(\mathbf{x}_m, t_m)$ *switching to the*

$(2k_i : 2k_j)$*-non-passable flow of the second kind on the boundary* $\overrightarrow{\partial\Omega}_{ij}$ *(or the switching bifurcation of the* $(2k_j : 2k_i)$*-passable flow from* $\overrightarrow{\partial\Omega}_{ij}$ to $\overleftarrow{\partial\Omega}_{ij}$*) occurs if and only if*

$$\left.\begin{aligned} G^{(s,j)}_{\partial\Omega_{ij}}(\mathbf{x}_m, t_{m\pm}, \mathbf{p}_j, \boldsymbol{\lambda}) = 0 \text{ for } s = 0, 1, \cdots, 2k_j - 1; \\ G^{(s,i)}_{\partial\Omega_{ij}}(\mathbf{x}_m, t_{m\mp}, \mathbf{p}_i, \boldsymbol{\lambda}) = 0 \text{ for } s = 0, 1, \cdots, 2k_i - 1; \end{aligned}\right\} \tag{2.129}$$

$$\left.\begin{aligned} &G^{(2k_i,i)}_{\partial\Omega_{ij}}(\mathbf{x}_m, t_{m\mp}, \mathbf{p}_i, \boldsymbol{\lambda}) = 0 \text{ and } G^{(2k_j,j)}_{\partial\Omega_{ij}}(\mathbf{x}_m, t_{m\pm}, \mathbf{p}_j, \boldsymbol{\lambda}) = 0, \text{ or} \\ &L^{(2k_i:2k_j)}_{ij}(\mathbf{x}_m, t_m, \mathbf{p}_i, \mathbf{p}_j, \boldsymbol{\lambda}) = 0, \text{ or} \\ &{}_{G\,\min}L^{(2k_i:2k_j)}_{ij}(t_m) = 0; \end{aligned}\right\} \tag{2.130}$$

$$\left.\begin{aligned} &\left.\begin{aligned} G^{(2k_i+1,i)}_{\partial\Omega_{ij}}(\mathbf{x}_m, t_{m\mp}, \mathbf{p}_i, \boldsymbol{\lambda}) < 0 \\ G^{(2k_j+1,j)}_{\partial\Omega_{ij}}(\mathbf{x}_m, t_{m\pm}, \mathbf{p}_j, \boldsymbol{\lambda}) > 0 \end{aligned}\right\} \text{ for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_j \\ &\left.\begin{aligned} G^{(2k_i+1,i)}_{\partial\Omega_{ij}}(\mathbf{x}_m, t_{m\mp}, \mathbf{p}_i, \boldsymbol{\lambda}) > 0 \\ G^{(2k_j+1,j)}_{\partial\Omega_{ij}}(\mathbf{x}_m, t_{m\pm}, \mathbf{p}_j, \boldsymbol{\lambda}) < 0 \end{aligned}\right\} \text{ for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_i. \end{aligned}\right\} \tag{2.131}$$

Proof The proof is the same as in the proof of Theorem 2.1 and Theorem 2.2. This theorem can be proved. □

Following the definitions in Definitions 2.28–2.35, the sliding and source fragmentation bifurcations can be similarly defined.

Definition 2.36 For a discontinuous dynamical system in Eq. (2.1), there is a point $\mathbf{x}^{(0)}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ at time t_m between two adjacent domains Ω_α ($\alpha = i, j$).

(i) The tangential bifurcation of the flow $\mathbf{x}^{(j)}(t)$ at point $(\mathbf{x}_m, t_m)$ on the boundary $\widetilde{\partial\Omega}_{ij}$ is termed *a fragmentation bifurcation of the non-passable flow of the first kind* (or called *a sliding fragmentation bifurcation*) if Eqs. (2.105) and (2.106) hold.

(ii) The tangential bifurcation of the flow $\mathbf{x}^{(i)}(t)$ with the $(2k_i)$th-order and $\mathbf{x}^{(j)}(t)$ of the $(2k_j)$th-order at point $(\mathbf{x}_m, t_m)$ on the boundary $\widetilde{\partial\Omega}_{ij}$ is termed *a fragmentation bifurcation of the* $(2k_i : 2k_j)$*-non-passable flow of the first kind* (or called *a* $(2k_i : 2k_j)$*-sliding fragmentation bifurcation*) if Eqs. (2.109) and (2.110) hold.

The necessary and sufficient conditions for the sliding fragmentation bifurcation of the non-passable flow of the first kind are given by Eqs. (2.107) and (2.108) with ${}_{G\,\max}L_{ij}(t_m)$ replacing ${}_{G\,\min}L_{ij}(t_m)$. Similarly, the necessary and sufficient conditions for the sliding fragmentation bifurcation of the $(2k_i : 2k_j)$-non-passable flow of the first kind are presented by Eqs. (2.111)–(2.113) with ${}_{G\,\max}L^{(2k_i:2k_j)}_{ij}(t_m)$ replacing ${}_{G\,\min}L^{(2k_i:2k_j)}_{ij}(t_m)$.

Definition 2.37 For a discontinuous dynamical system in Eq. (2.1), there is a point $\mathbf{x}^{(0)}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ at time t_m between two adjacent domains Ω_α $(\alpha = i, j)$.

(i) The tangential bifurcation of the flow $\mathbf{x}^{(j)}(t)$ at point $(\mathbf{x}_m, t_m)$ on the boundary $\widehat{\partial\Omega}_{ij}$ is termed *a fragmentation bifurcation of the non-passable flow of the second kind* (or called *a source fragmentation bifurcation*) if Eqs. (2.114) and (2.115) hold.

(ii) The tangential bifurcation of the flow $\mathbf{x}^{(i)}(t)$ with the $(2k_i)$th-order and $\mathbf{x}^{(j)}(t)$ of the $(2k_j)$th -order at point $(\mathbf{x}_m, t_m)$ on the boundary $\widehat{\partial\Omega}_{ij}$ is termed *the fragmentation bifurcation of the* $(2k_i : 2k_j)$*-non-passable flow of the second kind* (or called *a* $(2k_i : 2k_j)$*-source fragmentation bifurcation*) if Eqs. (2.118) and (2.119) hold.

The necessary and sufficient conditions for the source fragmentation bifurcation of the non-passable flow of the second kind are given by Eqs. (2.116) and (2.117) with ${}_{G\max}L_{ij}(t_m)$ replacing ${}_{G\min}L_{ij}(t_m)$. Similarly, the necessary and sufficient conditions for the sliding fragmentation bifurcation of the $(2k_i : 2k_j)$-non-passable flow of the second kind are presented by Eqs. (2.120)–(2.122) with ${}_{G\max}L_{ij}^{(2k_i:2k_j)}(t_m)$ replacing ${}_{G\min}L_{ij}^{(2k_i:2k_j)}(t_m)$.

Definition 2.38 For a discontinuous dynamical system in Eq. (2.1), there is a point $\mathbf{x}^{(0)}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ at time t_m between two adjacent domains Ω_α $(\alpha = i, j)$.

(i) The tangential bifurcation of the flow $\mathbf{x}^{(i)}(t)$ and $\mathbf{x}^{(j)}(t)$ at point $(\mathbf{x}_m, t_m)$ on the boundary $\widetilde{\partial\Omega}_{ij}$ (or $\widehat{\partial\Omega}_{ij}$) is termed *a switching bifurcation of the non-passable flow from* $\widetilde{\partial\Omega}_{ij}$ to $\widehat{\partial\Omega}_{ij}$ (or *from* $\widehat{\partial\Omega}_{ij}$ to $\widetilde{\partial\Omega}_{ij}$) if Eqs. (2.123) and (2.124) hold.

(ii) The tangential bifurcation of the flow $\mathbf{x}^{(i)}(t)$ with the $(2k_i)$th-order and $\mathbf{x}^{(j)}(t)$ with the $(2k_j)$th-order at point $(\mathbf{x}_m, t_m)$ on the boundary $\widetilde{\partial\Omega}_{ij}$ (or $\widehat{\partial\Omega}_{ij}$) is termed *a switching bifurcation of the* $(2k_i : 2k_j)$*-non-passable flow from* $\widetilde{\partial\Omega}_{ij}$ to $\widehat{\partial\Omega}_{ij}$ (or from $\widehat{\partial\Omega}_{ij}$ to $\widetilde{\partial\Omega}_{ij}$) if Eqs. (2.127) and (2.128) hold.

The necessary and sufficient conditions for the switching bifurcation of a non-passable flow from $\widetilde{\partial\Omega}_{ij}$ to $\widehat{\partial\Omega}_{ij}$ (or from $\widehat{\partial\Omega}_{ij}$ to $\widetilde{\partial\Omega}_{ij}$) are from Eqs. (2.125) and (2.126) with ${}_{G\max}L_{ij}(t_m)$ replacing ${}_{G\min}L_{ij}(t_m)$. However, the conditions for the switching bifurcation of the $(2k_i : 2k_j)$-non-passable flow from $\widetilde{\partial\Omega}_{ij}$ to $\widehat{\partial\Omega}_{ij}$ (or from $\widehat{\partial\Omega}_{ij}$ to $\widetilde{\partial\Omega}_{ij}$) the second kind are presented by Eqs. (2.129)–(2.131) with ${}_{G\max}L_{ij}^{(2k_i:2k_j)}(t_m)$ replacing ${}_{G\min}L_{ij}^{(2k_i:2k_j)}(t_m)$. The above conditions for the switching bifurcations of the $(2k_\alpha : 2k_\beta)$-flows are summarized in Table 2.1. In addition, the conditions for $(2k_\alpha : 2k_\beta - 1)$, $(2k_\alpha - 1 : 2k_\beta)$, and $(2k_\alpha - 1 : 2k_\beta)$-flows are presented in Tables 2.2–2.6, respectively. The following notations are used for simplicity.

$$L_{\alpha\beta}^{(m_\alpha:m_\beta)} \equiv L_{\alpha\beta}^{(m_\alpha:m_\beta)}(\mathbf{x}_m, t_m, \mathbf{p}_\alpha, \mathbf{p}_\beta, \boldsymbol{\lambda}) \tag{2.132}$$

Table 2.1 Sufficient and necessary conditions for $(2k_\alpha : 2k_\beta)$-switching bifurcations

$(2k_\alpha : 2k_\beta)$ passable flows $G_-^{(2k_\alpha)} > 0,\ G_+^{(2k_\beta)} > 0$ $L_{\alpha\beta}^{(2k_\alpha:2k_\beta)} > 0$	$\underset{G_-^{(2k_\beta)}=0,\ \text{or}\ L_{\alpha\beta}^{(2k_\alpha:2k_\beta)}=0}{\overset{G_+^{(2k_\beta)}=0,\ \text{or}\ L_{\alpha\beta}^{(2k_\alpha:2k_\beta)}=0}{\rightleftharpoons}}$ $\left.\begin{array}{l} G_\pm^{(2k_\beta+1)} > 0 \text{ for } \mathbf{n}_{\partial\Omega_{\alpha\beta}} \to \Omega_\beta; \\ G_\pm^{(2k_\beta+1)} < 0 \text{ for } \mathbf{n}_{\partial\Omega_{\alpha\beta}} \to \Omega_\alpha. \end{array}\right\}$	$(2k_\alpha : 2k_\beta)$ full-sink flows $G_-^{(2k_\alpha)} > 0,\ G_-^{(2k_\beta)} < 0$ $L_{\alpha\beta}^{(2k_\alpha:2k_\beta)} < 0$
$(2k_\alpha : 2k_\beta)$ passable flows $G_-^{(2k_\alpha)} > 0,\ G_+^{(2k_\beta)} > 0$ $L_{\alpha\beta}^{(2k_\alpha:2k_\beta)} > 0$	$\underset{G_+^{(2k_\alpha)}=0,\ \text{or}\ L_{\alpha\beta}^{(2k_\alpha:2k_\beta)}=0}{\overset{G_-^{(2k_\alpha)}=0,\ \text{or}\ L_{\alpha\beta}^{(2k_\alpha:2k_\beta)}=0}{\rightleftharpoons}}$ $\left.\begin{array}{l} G_\pm^{(2k_\alpha+1)} < 0 \text{ for } \mathbf{n}_{\partial\Omega_{\alpha\beta}} \to \Omega_\beta; \\ G_\pm^{(2k_\alpha+1)} > 0 \text{ for } \mathbf{n}_{\partial\Omega_{\alpha\beta}} \to \Omega_\alpha \end{array}\right\}$	$(2k_\alpha : 2k_\beta)$ full-source flows $G_+^{(2k_\alpha)} > 0,\ G_+^{(2k_\beta)} < 0$ $L_{\alpha\beta}^{(2k_\alpha:2k_\beta)} < 0$
$(2k_\alpha : 2k_\beta)$ passable flows $G_-^{(2k_\alpha)} > 0,\ G_+^{(2k_\beta)} > 0$ $L_{\alpha\beta}^{(2k_\alpha:2k_\beta)} > 0$	$\underset{G_+^{(2k_\alpha)}=0, G_-^{(2k_\beta)}=0;\ \text{or}\ L_{\alpha\beta}^{(2k_\alpha:2k_\beta)}=0}{\overset{G_-^{(2k_\alpha)}=0, G_+^{(2k_\beta)}=0;\ \text{or}\ L_{\alpha\beta}^{(2k_\alpha:2k_\beta)}=0,}{\rightleftharpoons}}$ $\left.\begin{array}{l} G_\pm^{(2k_\alpha+1)} < 0, G_\pm^{(2k_\beta+1)} > 0 \text{ for } \mathbf{n}_{\partial\Omega_{\alpha\beta}} \to \Omega_\beta; \\ G_\pm^{(2k_\alpha+1)} > 0, G_\pm^{(2k_\beta+1)} < 0 \text{ for } \mathbf{n}_{\partial\Omega_{\alpha\beta}} \to \Omega_\alpha \end{array}\right\}$	$(2k_\alpha : 2k_\beta)$ passable flows $G_+^{(2k_\alpha)} > 0,\ G_-^{(2k_\beta)} > 0$ $L_{\alpha\beta}^{(2k_\alpha:2k_\beta)} > 0$
$(2k_\alpha : 2k_\beta)$ full-sink flows $G_-^{(2k_\alpha)} > 0,\ G_-^{(2k_\beta)} < 0$ $L_{\alpha\beta}^{(2k_\alpha:2k_\beta)} < 0$	$\underset{G_+^{(2k_\alpha)}=0, G_+^{(2k_\beta)}=0;\ \text{or}\ L_{\alpha\beta}^{(2k_\alpha:2k_\beta)}=0}{\overset{G_-^{(2k_\alpha)}=0, G_-^{(2k_\beta)}=0;\ \text{or}\ L_{\alpha\beta}^{(2k_\alpha:2k_\beta)}=0,}{\rightleftharpoons}}$ $\left.\begin{array}{l} G_\pm^{(2k_\alpha+1)} < 0, G_\pm^{(2k_\beta+1)} > 0 \text{ for } \mathbf{n}_{\partial\Omega_{\alpha\beta}} \to \Omega_\beta; \\ G_\pm^{(2k_\alpha+1)} > 0, G_\pm^{(2k_\beta+1)} < 0 \text{ for } \mathbf{n}_{\partial\Omega_{\alpha\beta}} \to \Omega_\alpha \end{array}\right\}$	$(2k_\alpha : 2k_\beta)$ full-source flows $G_+^{(2k_\alpha)} > 0,\ G_+^{(2k_\beta)} < 0$ $L_{\alpha\beta}^{(2k_\alpha:2k_\beta)} < 0$

Table 2.2 Sufficient and necessary conditions for $(2k_\alpha : 2k_\beta - 1)$-switching bifurcations

$(2k_\alpha : 2k_\beta - 1)$ passable flows $G_-^{(2k_\alpha)} > 0,\ G_+^{(2k_\beta-1)} > 0$ $L_{\alpha\beta}^{(2k_\alpha:2k_\beta-1)} > 0$	$\underset{G_\pm^{(2k_\beta-1)}=0,\ G_\pm^{(2k_\beta)}>0}{\overset{G_+^{(2k_\beta-1)}=0, G_\pm^{(2k_\beta)}<0}{\rightleftharpoons}}$ $L_{\alpha\beta}^{(2k_\alpha:2k_\beta-1)} = 0$	$(2k_\alpha : 2k_\beta - 1)$ half-sink flows $G_-^{(2k_\alpha)} > 0,\ G_\pm^{(2k_\beta-1)} < 0$ $L_{\alpha\beta}^{(2k_\alpha:2k_\beta-1)} < 0$
$(2k_\alpha : 2k_\beta - 1)$ passable flows $G_-^{(2k_\alpha)} > 0,\ G_+^{(2k_\beta-1)} > 0$ $L_{\alpha\beta}^{(2k_\alpha:2k_\beta-1)} > 0$	$\underset{G_+^{(2k_\beta)}=0, G_\pm^{(2k_\beta-1)}=0, G_\pm^{(2k_\beta)}>0}{\overset{G_-^{(2k_\alpha)}=0,\ G_+^{(2k_\beta-1)}=0, G_\pm^{(2k_\beta)}<0}{\rightleftharpoons}}$ $L_{\alpha\beta}^{(2k_\alpha:2k_\beta-1)} = 0, G_\pm^{(2k_\alpha+1)} < 0$	$(2k_\alpha : 2k_\beta - 1)$ half-source flows $G_+^{(2k_\alpha)} < 0,\ G_\pm^{(2k_\beta-1)} < 0$ $L_{\alpha\beta}^{(2k_\alpha:2k_\beta-1)} < 0$
$(2k_\alpha : 2k_\beta - 1)$ passable flows $G_-^{(2k_\alpha)} > 0,\ G_+^{(2k_\beta-1)} > 0$ $L_{\alpha\beta}^{(2k_\alpha:2k_\beta-1)} > 0$	$\underset{G_+^{(2k_\alpha)}=0}{\overset{G_-^{(2k_\alpha)}=0}{\rightleftharpoons}}$ $L_{\alpha\beta}^{(2k_\alpha:2k_\beta-1)} = 0, G_\pm^{(2k_\alpha+1)} < 0$	$(2k_\alpha : 2k_\beta - 1)$ tangential flows $G_+^{(2k_\alpha)} < 0,\ G_\pm^{(2k_\beta-1)} > 0$ $L_{\alpha\beta}^{(2k_\alpha:2k_\beta-1)} > 0$
$(2k_\alpha : 2k_\beta - 1)$ half-sink flows $G_-^{(2k_\alpha)} > 0,\ G_\pm^{(2k_\beta-1)} < 0$ $L_{\alpha\beta}^{(2k_\alpha:2k_\beta-1)} < 0$	$\underset{G_+^{(2k_\alpha)}=0}{\overset{G_-^{(2k_\alpha)}=0}{\rightleftharpoons}}$ $L_{\alpha\beta}^{(2k_\alpha:2k_\beta)} = 0, G_\pm^{(2k_\alpha+1)} < 0$	$(2k_\alpha : 2k_\beta - 1)$ half-source flows $G_+^{(2k_\alpha)} < 0,\ G_\pm^{(2k_\beta-1)} < 0$ $L_{\alpha\beta}^{(2k_\alpha:2k_\beta-1)} < 0$

Table 2.3 Sufficient and necessary conditions for $(2k_\alpha : 2k_\beta - 1)$-switching bifurcations

$(2k_\alpha : 2k_\beta - 1)$ passable flows $G_-^{(2k_\alpha)} < 0,\ G_+^{(2k_\beta-1)} < 0$ $L_{\alpha\beta}^{(2k_\alpha:2k_\beta-1)} > 0$	$\underset{G_\pm^{(2k_\beta-1)} = 0, G_\pm^{(2k_\beta)} < 0}{\overset{G_+^{(2k_\beta-1)} = 0, G_\pm^{(2k_\beta)} > 0}{\rightleftharpoons}}$ $L_{\alpha\beta}^{(2k_\alpha:2k_\beta-1)} = 0$	$(2k_\alpha : 2k_\beta - 1)$ half-sink flows $G_-^{(2k_\alpha)} < 0,\ G_\pm^{(2k_\beta-1)} > 0$ $L_{\alpha\beta}^{(2k_\alpha:2k_\beta-1)} < 0$
$(2k_\alpha : 2k_\beta - 1)$ passable flows $G_-^{(2k_\alpha)} < 0,\ G_+^{(2k_\beta-1)} < 0$ $L_{\alpha\beta}^{(2k_\alpha:2k_\beta-1)} > 0$	$\underset{G_+^{(2k_\alpha)} = 0, G_\pm^{(2k_\beta-1)} = 0, G_\pm^{(2k_\beta)} < 0}{\overset{G_-^{(2k_\alpha)} = 0,\ G_+^{(2k_\beta-1)} = 0, G_\pm^{(2k_\beta)} > 0}{\rightleftharpoons}}$ $L_{\alpha\beta}^{(2k_\alpha:2k_\beta-1)} = 0, G_\pm^{(2k_\alpha+1)} > 0$	$(2k_\alpha : 2k_\beta - 1)$ half-source flows $G_+^{(2k_\alpha)} > 0,\ G_\pm^{(2k_\beta-1)} > 0$ $L_{\alpha\beta}^{(2k_\alpha:2k_\beta-1)} > 0$
$(2k_\alpha : 2k_\beta - 1)$ passable flows $G_-^{(2k_\alpha)} < 0,\ G_+^{(2k_\beta-1)} < 0$ $L_{\alpha\beta}^{(2k_\alpha:2k_\beta-1)} > 0$	$\underset{G_+^{(2k_\alpha)} = 0,}{\overset{G_-^{(2k_\alpha)} = 0,}{\rightleftharpoons}}$ $L_{\alpha\beta}^{(2k_\alpha:2k_\beta-1)} = 0, G_\pm^{(2k_\alpha+1)} > 0$	$(2k_\alpha : 2k_\beta - 1)$ tangential flows $G_+^{(2k_\alpha)} > 0,\ G_\pm^{(2k_\beta-1)} < 0$ $L_{\alpha\beta}^{(2k_\alpha:2k_\beta-1)} < 0$
$(2k_\alpha : 2k_\beta - 1)$ half-sink flows $G_-^{(2k_\alpha)} < 0,\ G_\pm^{(2k_\beta-1)} > 0$ $L_{\alpha\beta}^{(2k_\alpha:2k_\beta-1)} < 0$	$\underset{G_+^{(2k_\alpha)} = 0, G_+^{(2k_\beta)} = 0,}{\overset{G_-^{(2k_\alpha)} = 0, G_-^{(2k_\beta)} = 0}{\rightleftharpoons}}$ $L_{\alpha\beta}^{(2k_\alpha:2k_\beta)} = 0, G_\pm^{(2k_\alpha+1)} > 0$	$(2k_\alpha : 2k_\beta - 1)$ half-source flows $G_+^{(2k_\alpha)} > 0,\ G_\pm^{(2k_\beta-1)} > 0$ $L_{\alpha\beta}^{(2k_\alpha:2k_\beta-1)} > 0$

Table 2.4 Sufficient and necessary conditions for $(2k_\alpha - 1 : 2k_\beta)$-switching bifurcations

$(2k_\alpha - 1 : 2k_\beta)$ tangential flows $G_\pm^{(2k_\alpha-1)} < 0,\ G_+^{(2k_\beta)} > 0$ $L_{\alpha\beta}^{(2k_\alpha-1:2k_\beta)} < 0$	$\underset{G_\pm^{(2k_\beta-1)} = 0, G_-^{(2k_\beta+1)} > 0}{\overset{G_+^{(2k_\beta)} = 0, G_\pm^{(2k_\beta+1)} < 0}{\rightleftharpoons}}$ $L_{\alpha\beta}^{(2k_\alpha-1:2k_\beta)} = 0$	$(2k_\alpha - 1 : 2k_\beta)$ tangential flows $G_\pm^{(2k_\alpha-1)} < 0,\ G_-^{(2k_\beta)} < 0$ $L_{\alpha\beta}^{(2k_\alpha-1:2k_\beta)} > 0$
$(2k_\alpha - 1 : 2k_\beta)$ tangential flows $G_\pm^{(2k_\alpha-1)} < 0,\ G_+^{(2k_\beta)} > 0$ $L_{\alpha\beta}^{(2k_\alpha-1:2k_\beta)} < 0$	$\underset{G_\pm^{(2k_\alpha-1)} = 0, G_-^{(2k_\beta)} = 0; G_-^{(2k_\alpha)} < 0, G_\pm^{(2k_\beta+1)} > 0}{\overset{G_\pm^{(2k_\alpha-1)} = 0,\ G_+^{(2k_\beta)} = 0; G_+^{(2k_\alpha)} > 0, G_\pm^{(2k_\beta+1)} < 0}{\rightleftharpoons}}$ $L_{\alpha\beta}^{(2k_\alpha-1:2k_\beta)} = 0,$	$(2k_\alpha - 1 : 2k_\beta)$ half-sink flows $G_\pm^{(2k_\alpha-1)} > 0,\ G_-^{(2k_\beta)} < 0$ $L_{\alpha\beta}^{(2k_\alpha-1:2k_\beta)} < 0$
$(2k_\alpha - 1 : 2k_\beta)$ tangential flows $G_\pm^{(2k_\alpha-1)} < 0,\ G_+^{(2k_\beta)} > 0$ $L_{\alpha\beta}^{(2k_\alpha-1:2k_\beta)} < 0$	$\underset{G_\pm^{(2k_\alpha-1)} = 0, G_-^{(2k_\alpha)} < 0}{\overset{G_\pm^{(2k_\alpha-1)} = 0,\ G_+^{(2k_\alpha)} > 0}{\rightleftharpoons}}$ $L_{\alpha\beta}^{(2k_\alpha-1:2k_\beta)} = 0$	$(2k_\alpha - 1 : 2k_\beta)$ half-source flows $G_\pm^{(2k_\alpha-1)} > 0,\ G_+^{(2k_\beta)} > 0$ $L_{\alpha\beta}^{(2k_\alpha-1:2k_\beta)} > 0$
$(2k_\alpha - 1 : 2k_\beta)$ half-sink flows $G_\pm^{(2k_\alpha-1)} > 0,\ G_-^{(2k_\beta)} < 0$ $L_{\alpha\beta}^{(2k_\alpha-1:2k_\beta)} < 0$	$\underset{G_+^{(2k_\beta)} = 0; G_\pm^{(2k_\beta+1)} < 0}{\overset{G_-^{(2k_\beta)} = 0;\ G_\pm^{(2k_\beta+1)} > 0}{\rightleftharpoons}}$ $L_{\alpha\beta}^{(2k_\alpha:2k_\beta)} = 0,$	$(2k_\alpha - 1 : 2k_\beta)$ half-source flows $G_\pm^{(2k_\alpha-1)} > 0,\ G_+^{(2k_\beta)} > 0$ $L_{\alpha\beta}^{(2k_\alpha-1:2k_\beta)} > 0$

Table 2.5 Sufficient and necessary conditions for $(2k_\alpha - 1 : 2k_\beta)$-switching bifurcations

$(2k_\alpha - 1 : 2k_\beta)$ tangential flows $G_\pm^{(2k_\alpha-1)} > 0,\ G_-^{(2k_\beta)} > 0$ $L_{\alpha\beta}^{(2k_\alpha-1:2k_\beta)} > 0$	$\underset{G_\pm^{(2k_\beta)}=0;G_-^{(2k_\beta+1)}>0}{\overset{G_+^{(2k_\beta)}=0;G_\pm^{(2k_\beta+1)}<0}{\rightleftharpoons}}$ $L_{\alpha\beta}^{(2k_\alpha-1:2k_\beta)} = 0$	$(2k_\alpha - 1 : 2k_\beta)$ tangential flows $G_\pm^{(2k_\alpha-1)} > 0,\ G_+^{(2k_\beta)} < 0$ $L_{\alpha\beta}^{(2k_\alpha-1:2k_\beta)} < 0$
$(2k_\alpha - 1 : 2k_\beta)$ tangential flows $G_\pm^{(2k_\alpha-1)} > 0,\ G_+^{(2k_\beta)} > 0$ $L_{\alpha\beta}^{(2k_\alpha-1:2k_\beta)} > 0$	$\underset{G_\pm^{(2k_\alpha-1)}=0,G_-^{(2k_\beta)}=0;G_-^{(2k_\alpha)}>0,G_\pm^{(2k_\beta+1)}>0}{\overset{G_\pm^{(2k_\alpha-1)}=0,\ G_+^{(2k_\beta)}=0;G_+^{(2k_\alpha)}<0,G_\pm^{(2k_\beta+1)}<0}{\rightleftharpoons}}$ $L_{\alpha\beta}^{(2k_\alpha:2k_\beta-1)} = 0$	$(2k_\alpha - 1 : 2k_\beta)$ half-source flows $G_\pm^{(2k_\alpha-1)} < 0,\ G_+^{(2k_\beta)} < 0$ $L_{\alpha\beta}^{(2k_\alpha-1:2k_\beta)} > 0$
$(2k_\alpha - 1 : 2k_\beta)$ tangential flows $G_\pm^{(2k_\alpha-1)} > 0,\ G_-^{(2k_\beta)} > 0$ $L_{\alpha\beta}^{(2k_\alpha-1:2k_\beta)} > 0$	$\underset{G_\pm^{(2k_\alpha-1)}=0,G_-^{(2k_\alpha)}>0}{\overset{G_\pm^{(2k_\alpha-1)}=0,\ G_+^{(2k_\alpha)}<0}{\rightleftharpoons}}$ $L_{\alpha\beta}^{(2k_\alpha:2k_\beta-1)} = 0$	$(2k_\alpha - 1 : 2k_\beta)$ half-sink flows $G_\pm^{(2k_\alpha-1)} < 0,\ G_-^{(2k_\beta)} > 0$ $L_{\alpha\beta}^{(2k_\alpha-1:2k_\beta)} < 0$
$(2k_\alpha - 1 : 2k_\beta)$ half-source flows $G_\pm^{(2k_\alpha-1)} < 0,\ G_+^{(2k_\beta)} < 0$ $L_{\alpha\beta}^{(2k_\alpha-1:2k_\beta)} > 0$	$\underset{G_+^{(2k_\beta)}=0;G_\pm^{(2k_\beta+1)}<0}{\overset{G_-^{(2k_\beta)}=0;G_\pm^{(2k_\beta+1)}>0}{\rightleftharpoons}}$ $L_{\alpha\beta}^{(2k_\alpha:2k_\beta)} = 0,$	$(2k_\alpha - 1 : 2k_\beta)$ half-sink flows $G_\pm^{(2k_\alpha-1)} < 0,\ G_-^{(2k_\beta)} > 0$ $L_{\alpha\beta}^{(2k_\alpha-1:2k_\beta)} < 0$

Table 2.6 Sufficient and necessary conditions for $(2k_i - 1 : 2k_j - 1)$-switching bifurcations

$(2k_i - 1 : 2k_j - 1)$ double tangential flows $G_\pm^{(2k_i-1)} < 0,\ G_\pm^{(2k_j-1)} > 0$ $L_{ij}^{(2k_i-1:2k_j-1)} < 0$	$\underset{G_\pm^{(2k_i-1)}=0,G_\pm^{(2k_i)}<0;G_\pm^{(2k_j-1)}=0,G_\pm^{(2k_j)}>0}{\overset{G_\pm^{(2k_i-1)}=0,G_\pm^{(2k_i)}>0;G_\pm^{(2k_j-1)}=0,G_\pm^{(2k_j)}<0}{\rightleftharpoons}}$ $L_{ij}^{(2k_i-1:2k_i-1)} = 0$	$(2k_i - 1 : 2k_j - 1)$ double inaccessible flows $G_\pm^{(2k_i-1)} > 0,\ G_\pm^{(2k_j-1)} < 0$ $L_{ij}^{(2k_i-1:2k_j-1)} < 0$
$(2k_i - 1 : 2k_j - 1)$ double tangential flows $G_\pm^{(2k_i-1)} < 0,\ G_\pm^{(2k_j-1)} > 0$ $L_{ij}^{(2k_i-1:2k_j-1)} < 0$	$\underset{G_\pm^{(2k_j-1)}=0,G_\pm^{(2k_j)}>0}{\overset{G_\pm^{(2k_j-1)}=0,G_\pm^{(2k_j)}<0}{\rightleftharpoons}}$ $L_{ij}^{(2k_i-1:2k_i-1)} = 0$	$(2k_i - 1 : 2k_j - 1)$ single tangential flows in Ω_i $G_\pm^{(2k_i-1)} < 0,\ G_\pm^{(2k_j-1)} < 0$ $L_{ij}^{(2k_i-1:2k_j-1)} > 0$
$(2k_i - 1 : 2k_j - 1)$ double tangential flows $G_\pm^{(2k_i-1)} < 0,\ G_\pm^{(2k_j-1)} > 0$ $L_{ij}^{(2k_i-1:2k_j-1)} < 0$	$\underset{G_\pm^{(2k_i-1)}=0,G_\pm^{(2k_i)}<0}{\overset{G_\pm^{(2k_i-1)}=0,G_\pm^{(2k_i)}>0}{\rightleftharpoons}}$ $L_{ij}^{(2k_i-1:2k_i-1)} = 0$	$(2k_i - 1 : 2k_j - 1)$ single tangential flows in Ω_j $G_\pm^{(2k_i-1)} > 0,\ G_\pm^{(2k_j-1)} > 0$ $L_{ij}^{(2k_i-1:2k_j-1)} > 0$
$(2k_i - 1 : 2k_j - 1)$ single tangential flows in Ω_i $G_\pm^{(2k_i-1)} < 0,\ G_\pm^{(2k_j-1)} < 0$ $L_{ij}^{(2k_i-1:2k_j-1)} > 0$	$\underset{G_\pm^{(2k_i-1)}=0,G_\pm^{(2k_i)}<0;G_\pm^{(2k_j-1)}=0,G_\pm^{(2k_j)}>0}{\overset{G_\pm^{(2k_i-1)}=0,G_\pm^{(2k_i)}>0;G_\pm^{(2k_j-1)}=0,G_\pm^{(2k_j)}<0}{\rightleftharpoons}}$ $L_{ij}^{(2k_i-1:2k_i-1)} = 0$	$(2k_i - 1 : 2k_j - 1)$ single tangential flows in Ω_j $G_\pm^{(2k_i-1)} > 0,\ G_\pm^{(2k_j-1)} > 0$ $L_{ij}^{(2k_i-1:2k_j-1)} > 0$

$$
\begin{aligned}
G_{\pm}^{(m_\alpha,\alpha)} &\equiv G_{\partial\Omega_{\alpha\beta}}^{(m_\alpha,\alpha)}(\mathbf{x}_m, t_{m\pm}, \mathbf{p}_\alpha, \boldsymbol{\lambda}) \\
G_{-}^{(m_\alpha,\alpha)} &\equiv G_{\partial\Omega_{\alpha\beta}}^{(m_\alpha,\alpha)}(\mathbf{x}_m, t_{m-}, \mathbf{p}_\alpha, \boldsymbol{\lambda}) \\
G_{+}^{(m_\alpha,\alpha)} &\equiv G_{\partial\Omega_{\alpha\beta}}^{(m_\alpha,\alpha)}(\mathbf{x}_m, t_{m+}, \mathbf{p}_\alpha, \boldsymbol{\lambda})
\end{aligned}
\tag{2.133}
$$

Because the concept of the imaginary flow is introduced, the switching bifurcations of the $(2k_\alpha : 2k_\beta - 1)$, $(2k_\alpha - 1 : 2k_\beta)$, and $(2k_\alpha - 1 : 2k_\beta)$-flows can follow the discussion on the switching bifurcation of the $(2k_\alpha : 2k_\beta)$-flows. The corresponding necessary and sufficient conditions for such flows can be obtained. For $(2k_\alpha : 2k_\beta - 1)$, $(2k_\alpha - 1 : 2k_\beta)$, and $(2k_\alpha - 1 : 2k_\beta)$-flows, the switching bifurcations between a passable flow and half-non-passable flow, and between a passable flow and single tangential flow are presented. In addition, the switching bifurcations between a half-non-passable flow to a single tangential flow, and between a double tangential flow and a double inaccessible flow are given.

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Chapter 3
Single Constraint Synchronization

In this chapter, the synchronization of two or more dynamical systems to specific constraints is introduced, which is different from the traditional synchronization of two dynamical systems. For such synchronization, Lyapunov stability method cannot be adopted. The synchronization, desynchronization, and penetration of multiple dynamical systems to a specific constraint are discussed from the theory of discontinuous dynamical systems, and the necessary and sufficient conditions for such synchronicity are presented.

3.1 Introduction to Synchronization

As in Luo [1], consider two dynamic systems as

$$\dot{\mathbf{x}}^{(r)} = \mathbf{F}^{(r)}(\mathbf{x}^{(r)}, t, \mathbf{p}^{(r)}) \in \mathscr{R}^{n_r} \tag{3.1}$$

and

$$\dot{\mathbf{x}}^{(s)} = \mathbf{F}^{(s)}(\mathbf{x}^{(s)}, t, \mathbf{p}^{(s)}) \in \mathscr{R}^{n_s} \tag{3.2}$$

For $\sigma = \{r, s\}$, $\mathbf{F}^{(\sigma)} = (F_1^{(\sigma)}, F_2^{(\sigma)}, \cdots, F_{n_\sigma}^{(\sigma)})^{\mathrm{T}}$, $\mathbf{x}^{(\sigma)} = (x_1^{(\sigma)}, x_2^{(\sigma)}, \cdots, x_{n_\sigma}^{(\sigma)})^{T}$, and parameter vector $\mathbf{p}^{(\sigma)} = (p_1^{(\sigma)}, p_2^{(\sigma)}, \cdots, p_{k_\sigma}^{(\sigma)})^{\mathrm{T}}$. The vector functions $\mathbf{F}^{(\sigma)}$ can be time-dependent or time-independent. Consider a time interval $I_{12} \equiv (t_1, t_2) \in \mathscr{R}$ and domains $U_{\mathbf{x}^{(\sigma)}} \subseteq \mathscr{R}^{n_\sigma}$ $(\sigma = \{\alpha, \beta\})$. $(t_0, \mathbf{x}_0^{(\sigma)}) \in I_{12} \times U_{\mathbf{x}^{(\sigma)}}$ is initial condition, and the corresponding flows of the two systems are $\mathbf{x}^{(\sigma)}(t) = \mathbf{\Phi}(t, \mathbf{x}_0^{(\sigma)}, t_0, \mathbf{p}^{(\sigma)})$ for $(t, \mathbf{x}^{(\sigma)}) \in I_{12} \times U_{\mathbf{x}^{(\sigma)}}$. The semigroup properties of two flows hold. To discuss the synchronization of the two systems in Eqs. (3.1) and (3.2), the concepts of the slave and master systems are introduced herein.

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Definition 3.1 A system in Eq. (3.2) is called a *master* system if its flow $\mathbf{x}^{(r)}(t)$ is independent. A system in Eq. (3.1) is called a *slave* system of the master system if its flow $\mathbf{x}^{(s)}(t)$ is constrained by a flow $\mathbf{x}^{(r)}(t)$ of the master system.

From the foregoing definition, a *slave* system is constrained by a *master* system via a specific condition, which means that a slave system will be controlled by a master system under a specific constraint. Such a phenomenon is called the synchronization of the slave and master systems under such a specific condition. To make this concept clear, a definition is given as follows.

Definition 3.2 If a flow $\mathbf{x}^{(s)}(t)$ of a slave system in Eq. (3.1) is constrained by a flow $\mathbf{x}^{(r)}(t)$ of a *master* system in Eq. (3.2) through

$$\varphi(\mathbf{x}^{(r)}(t), \mathbf{x}^{(s)}(t), t, \boldsymbol{\lambda}) = 0, \quad \boldsymbol{\lambda} \in \mathscr{R}^{n_0} \tag{3.3}$$

for time $t \in [t_{m_1}, t_{m_2}]$, then the slave system is said to be *synchronized* with the master system in the sense of Eq. (3.3) for time $t \in [t_{m_1}, t_{m_2}]$, also called an $(n_r : n_s)$-dimensional synchronization of the slave and master systems in the sense of Eq. (3.3). Four special cases are given as follows.

(i) If $t_{m_2} \to \infty$, the slave system is said to be absolutely synchronized with the master system in the sense of Eq. (3.3) for time $t \in [t_{m_1}, \infty)$.
(ii) If $t_{m_1} \to \infty$, the slave system is said to *be asymptotically synchronized* with the master system in the sense of Eq. (3.3).
(iii) For $n_r = n_s$, such a synchronization of the slave and master systems is called an *equidimensional* system synchronization in the sense of Eq. (3.3) for time $t \in [t_{m_1}, t_{m_2}]$.
(iv) For $n_r = n_s$, such a synchronization of the slave and master systems is called an *absolute*, equidimensional system synchronization in the sense of Eq. (3.3) for time $t \in [t_{m_1}, \infty)$.

If $n_r \neq n_s$, the $(n_r : n_s)$-dimensional synchronization is called a non-equidimensional system synchronization. It indicates that the dimension number of a slave system can be less or more than one of the master system. Thus, it is not necessary to require the slave and master systems have the same dimensions for synchronization. Under a certain rule in Eq. (3.3), it is interesting that a slave system can follow another completely different master system to synchronize. From the foregoing definition, it can be seen that a slave system is synchronized with a master system under a constraint condition. In fact, constraints for such a synchronization phenomenon can be more than one. In other words, a slave system is synchronized with a master system under multiple constraints. Thus, the synchronization of a slave system with a master system under multiple constraints is defined.

Definition 3.3 An n_s-dimensional slave system in Eq. (3.1) is called to be synchronized with an n_r-dimensional master system in Eq. (3.2) of the $(n_r : n_s; l)$-*type* (or an $(n_r : n_s; l)$-*synchronization*) if there are l-linearly independent functions

$\varphi_j(\mathbf{x}^{(r)}(t), \mathbf{x}^{(s)}(t), t, \boldsymbol{\lambda}_j)$ ($j \in \mathscr{L}$ and $\mathscr{L} = \{1, 2, \cdots, l\}$) to make two flows $\mathbf{x}^{(r)}(t)$ and $\mathbf{x}^{(s)}(t)$of the master and slave systems satisfy

$$\varphi_j(\mathbf{x}^{(r)}(t), \mathbf{x}^{(s)}(t), t, \boldsymbol{\lambda}_j) = 0, \quad \boldsymbol{\lambda}_j \in \mathscr{R}^{n_j} \text{ and } j \in \mathscr{L} \tag{3.4}$$

for time $t \in [t_{m_1}, t_{m_2}]$. Eight special cases are given as follows:

(i) If $t_{m_2} \to \infty$, the slave system is said to be *absolutely* synchronized of the $(n_r\!: n_s; l)$*-type* with the master system (or an $(n_r : n_s; l)$-absolute synchronization) in the sense of Eq. (3.4) for time $t \in [t_{m_1}, \infty)$.
(ii) If $t_{m_1} \to \infty$, the slave system is said to be *asymptotically* synchronized of the $(n_r : n_s; l)$*-type* with the master system (*or* an $(n_r : n_s; l)$*-asymptotic synchronization*) in the sense of Eq. (3.4).
(iii) For $l = n_s$, the slave system is said to be *completely* synchronized of the $(n_r\!: n_s; n_s)$*-type* with the master system (*or an* $(n_r : n_s; n_s)$*-complete synchronization*) in the sense of Eq. (3.4) for time $t \in [t_{m_1}, t_{m_2}]$.
(iv) For $l = n_s$ and $t_{m_2} \to \infty$, the synchronization of the slave and master systems is called *an* $(n_r : n_s; n_s)$*-absolute, complete synchronization* in the sense of Eq. (3.4) for time $t \in [t_{m_1}, \infty)$.
(v) If $n_r = n_s = n > l$, the synchronization of the slave and master systems is called an equidimensional system synchronization (or an $(n : n; l)$synchronization) in the sense of Eq. (3.4) for time $t \in [t_{m_1}, t_{m_2}]$.
(vi) If $n_r = n_s = n > l$ and $t_{m_1} \to \infty$, the synchronization of the slave and master systems is called an equidimensional, $(n : n; l)$-absolute synchronization in the sense of Eq. (3.4) for time $t \in [t_{m_1}, \infty)$.
(vii) If $n_r = n_s = n = l$, the synchronization of the slave and master systems is called an equidimensional, complete synchronization (usually called *a synchronization*) in the sense of Eq. (3.4) for time $t \in [t_{m_1}, t_{m_2}]$.
(viii) If $n_r = n_s = n = l$ and $t_{m_2} \to \infty$, the synchronization of the slave and master systems is called an equidimensional, absolute, complete synchronization (or called an *absolute* synchronization) in the sense of Eq. (3.4) for time $t \in [t_{m_1}, \infty)$.

In the foregoing definition, if the l-nonlinear equations are linearly independent, then there is a set of constants k_j and only $k_j = 0$ for all $j \in \mathscr{L}$ exists to make the following equation hold for all the domains and time,

$$\sum\nolimits_{j=1}^{l} k_j \varphi_j(\mathbf{x}^{(r)}(t), \mathbf{x}^{(s)}(t), t, \boldsymbol{\lambda}_j) = 0. \tag{3.5}$$

In addition, the independence of functions $\varphi_j(\mathbf{x}^{(r)}(t), \mathbf{x}^{(s)}(t), t, \boldsymbol{\lambda}_j)$ (for all $j \in \mathscr{L}$) is checked through the corresponding normal vectors. The normal vector of $\varphi_j(\mathbf{x}^{(r)}(t),\mathbf{x}^{(s)}(t), t, \boldsymbol{\lambda}_j)$ is computed by

$$\mathbf{n}_{\varphi_j} = \nabla \varphi_j(\mathbf{x}^{(r)}(t), \mathbf{x}^{(s)}(t), t, \boldsymbol{\lambda}_j) = \left(\frac{\partial \varphi_j}{\partial \mathbf{x}^{(r)}}, \frac{\partial \varphi_j}{\partial \mathbf{x}^{(s)}} \right)^{\mathrm{T}}. \tag{3.6}$$

For all domains and time, if all the normal vectors $\mathbf{n}_{\varphi_j}$ $(j \in \mathscr{L})$ are linearly independent, i.e.,

$$\sum\nolimits_{j=1}^{l} k_j \mathbf{n}_{\varphi_j} = 0 \quad \text{only if } k_j = 0 \text{ for all } j \in \mathscr{L}. \tag{3.7}$$

then the functions $\varphi_j(\mathbf{x}^{(r)}(t), \mathbf{x}^{(s)}(t), t, \boldsymbol{\lambda}_j)$are linearly independent.

The foregoing definition tells that the slave and master systems are synchronized under l-constraints whatever the state-space dimension of the slave system is higher or lower than the master system. For $l < n_s$, the l-variables of the n-state variables of the slave system can be expressed by the n-state variables of the master system via the l-constraints. Select any l-variables $x_{[j]}$ and the rest $(n_s - l)$ variables $x_{[k]}$ of the n_s-state variables, i.e.,

$$\begin{aligned} &x_{[j]}^{(s)} \in \{x_i, i = 1, 2, \cdots, n_s\} \quad \text{for } j = 1, 2, \cdots, l \\ &x_{(k)} \in \{x_i, i = 1, 2, \cdots, n_s\} \quad \text{for } k = l+1, l+2, \cdots, n_s \end{aligned} \tag{3.8}$$

From Eq. (3.4), due to the linear independence of functions $\varphi_j(\mathbf{x}^{(r)}(t), \mathbf{x}^{(s)}(t), t, \boldsymbol{\lambda}_j)$ $(j = 1, 2, \cdots, l)$, the constraint conditions give

$$x_{[j]} = f_{[j]}(\mathbf{x}^{(s)}, x_{(l+1)}, x_{(l+2)}, \cdots, x_{(n_s)}, \boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2 \cdots, \boldsymbol{\lambda}_l) \quad \text{for } j \in \mathscr{L}. \tag{3.9}$$

In this case, the state variables $x_{[j]}$ for $j \in \mathscr{L}$ can be said to be synchronized with the master system in the conditions of Eq. (3.4). The subscripts $[\cdot]$ and $(\cdot)$ of the state variables of the slave systems stand for the *synchronizable* and *non-synchronizable* variables to the master systems, respectively. If $l = 1$, this definition is reduced to Definition 3.2 and $(n_r : n_s; 1) \equiv (n_r : n_s)$, the $(n_r : n_s; l)$-synchronization reduces to the $(n_r : n_s)$-synchronization. However, for $l = n_s$, the n_s-linearly independent conditions constrain the responses of the master and slave flows in the n_r-dimensional systems. Thus, the n_s-components of the slave flow can be completely determined by the n_r-components of a flow in the master system. Therefore, for the complete synchronization of the slave and master systems, a flow of the slave system is completely controlled by the master system through the constraint conditions in Eq. (3.4). For $l > n_s$, the slave system is overconstrained by the master system. Such a case will be discussed later. For $n_r = n_s = n = l$, an equi-dimensional, complete synchronization of the slave and master systems is obtained. For this case, n-components of a flow in the slave system are controlled by the n-components of a flow in the master system through the n-constraint equations in Eq. (3.4). Because the n-constraint equations in Eq. (3.4) are linearly independent, the determinant of the Jacobian matrix of functions in Eq. (3.4) in neighborhood of the master flow $\mathbf{x}^{(r)}$ is nonzero. Therefore, there is a one-to-one relation between the slave and master flows $\mathbf{x}^{(s)}$ and $\mathbf{x}^{(r)}$. It implies that the slave flow is completely controlled by the master flow. From the above discussion, one obtains

$$\begin{aligned}
&\mathbf{x}^{(s)}(t) = \mathbf{h}(\mathbf{x}^{(r)}(t), \boldsymbol{\lambda}) \quad \text{or} \\
&x_i^{(s)}(t) = h_i(\mathbf{x}^{(r)}(t), \boldsymbol{\lambda}) \quad \text{for } i = 1, 2, \cdots, n.
\end{aligned} \tag{3.10}$$

Introduce a set of new variables with n-linear, independent relations between the slave and master systems. So one has

$$\begin{aligned}
&\mathbf{z}(t) = \mathbf{x}^{(s)}(t) - \mathbf{B}\mathbf{x}^{(r)}(t) = \mathbf{h}(\mathbf{x}^{(r)}) - \mathbf{B}\mathbf{x}^{(r)} \quad \text{or} \\
&z_i(t) = x_i^{(s)}(t) - b_i x_i^{(r)}(t) = h_i(\mathbf{x}^{(r)}) - b_i x_i^{(r)}(t) \quad \text{for } i = 1, 2, \cdots, n
\end{aligned} \tag{3.11}$$

where a constant diagonal matrix $\mathbf{B} = diag(b_1, b_2, \cdots, b_n)$. In recent researches on the synchronization of two systems, one likes to make $z_i(t) \to 0$ for $t \to t_{m_1}$ and $z_i(t) = 0$ for $t \in [t_{m_1}, t_{m_2}]$, from which the slave and master system are synchronized. To achieve such synchronization, the fixed points of $b_i x_i^{(r)}(t) = h_i(\mathbf{x}^{(r)})$ for $i = 1, 2, \cdots, n$ can be determined and independent of time. Such a concept can be extended to such linear synchronization, i.e., for $z_i(t) \to c_i$ (constant) for $t \to t_m$ and $z_i(t) = c_i$ for $t \in [t_{m_1}, t_{m_2}]$. The definition is given as follows:

Definition 3.4 For the slave and master in Eqs. (3.1) and (3.2) with $n_r = n_s = n$, if the slave and master flows satisfy

$$\mathbf{x}^{(s)}(t) - \mathbf{B}\mathbf{x}^{(r)}(t) = \mathbf{c} \tag{3.12}$$

with a constant diagonal matrix $\mathbf{B} = diag(b_1, b_2, \cdots, b_n)$ and a constant vector $\mathbf{c} = (c_1, c_2, \cdots, c_n)^{\mathrm{T}}$ for $t \in [t_{m_1}, t_{m_2}]$, then the slave and master systems are equidimensionally synchronized in such a linear sense. If $t_{m+1} \to \infty$, the synchronization of the slave and master systems is absolutely and equidimensionally synchronized in the linear sense for time $t \in [t_{m_1}, \infty)$. Three important synchronizations are also given as follows:

(i) If $\mathbf{c} = \mathbf{0}$ and $b_i = 1$ $(i = 1, 2, \cdots, n)$, the synchronization of the slave and master systems is called an identical synchronization.
(ii) If $\mathbf{c} = \mathbf{0}$ and $b_i = -1$ $(i = 1, 2, \cdots, n)$, the synchronization of the slave and master systems is called an antisymmetric synchronization.
(iii) If $\mathbf{c} = \mathbf{0}$ and $b_i \in \{1, -1\}$ $(i = 1, 2, \cdots, n)$, the synchronization of the slave and master systems is called a mixed, identical and antisymmetric synchronization.

To extend the above idea, new variables are introduced as

$$\begin{aligned}
&z_j = \varphi_j(\mathbf{x}^{(r)}(t), \mathbf{x}^{(s)}(t), t, \boldsymbol{\lambda}_j), \quad j \in \mathscr{L} \\
&\mathbf{z} = \boldsymbol{\varphi}(\mathbf{x}^{(r)}(t), \mathbf{x}^{(s)}(t), t, \boldsymbol{\lambda})
\end{aligned} \tag{3.13}$$

If $z_j = c_j$ (const) or $z_j = 0$, Eq. (3.13) can be used as the constraint condition in Eq. (3.4). If the slave and master systems are not synchronized, the new variables

$(z_j \neq c_j,\ j = 1, 2, \cdots, l)$ will change with time t. The corresponding time-change rate is given by

$$\begin{aligned}
\dot{z}_j &= D\varphi_j(\mathbf{x}^{(r)}(t), \mathbf{x}^{(s)}(t), t, \boldsymbol{\lambda}_j) = \frac{\partial \varphi_j}{\partial \mathbf{x}^{(r)}} \dot{\mathbf{x}}^{(\alpha)} + \frac{\partial \varphi_j}{\partial \mathbf{x}^{(s)}} \dot{\mathbf{x}}^{(\beta)} + \frac{\partial \varphi_j}{\partial t} \\
&= \frac{\partial \varphi_j}{\partial \mathbf{x}^{(r)}} \mathbf{F}^{(r)} + \frac{\partial \varphi_j}{\partial \mathbf{x}^{(s)}} \mathbf{F}^{(s)} + \frac{\partial \varphi_j}{\partial t}, \\
\dot{\mathbf{z}} &= D\boldsymbol{\varphi}(\mathbf{x}^{(r)}(t), \mathbf{x}^{(s)}(t), t, \boldsymbol{\lambda}) = \frac{\partial \boldsymbol{\varphi}}{\partial \mathbf{x}^{(r)}} \dot{\mathbf{x}}^{(r)} + \frac{\partial \boldsymbol{\varphi}}{\partial \mathbf{x}^{(s)}} \dot{\mathbf{x}}^{(s)} + \frac{\partial \boldsymbol{\varphi}}{\partial t} \\
&= \frac{\partial \boldsymbol{\varphi}}{\partial \mathbf{x}^{(r)}} \mathbf{F}^{(r)} + \frac{\partial \boldsymbol{\varphi}}{\partial \mathbf{x}^{(s)}} \mathbf{F}^{(s)} + \frac{\partial \boldsymbol{\varphi}}{\partial t}.
\end{aligned} \tag{3.14}$$

For simplicity, $D\varphi_j = \varphi_j^{(1)}$ and $D^r\varphi_j = \varphi_j^{(r)}$ are adopted from now on. If the slave and master systems are continuous, the time-change rate of the new variables for the constraint conditions in Eq. (3.4) should be zero, i.e., $\dot{z}_j = 0$ $(j \in \mathscr{L})$ or $\dot{\mathbf{z}} = \mathbf{0} \in \mathscr{R}^l$. However, if the slave and master systems are discontinuous to the constraint conditions, the time-change rate of the new variables for the constraint conditions in Eq. (3.4) may not be zero. To investigate the synchronization, the constraints are considered as boundaries in discontinuous dynamical systems.

The slave and master flows $\mathbf{x}^{(s)}(t)$ and $\mathbf{x}^{(r)}(t)$ are determined by differential equations in Eqs. (3.1) and (3.2). Suppose at least there is a point $\mathbf{x}_m$ at time t_m to satisfy the constraint condition in Eq. (3.3), i.e.,

$$z_m = \varphi(\mathbf{x}_m^{(r)}, \mathbf{x}_m^{(s)}, t_m, \boldsymbol{\lambda}) = 0 \tag{3.15}$$

For $t > t_m$, the synchronization between the slave and master systems requires the slave and master flows to satisfy the constraint condition in Eq. (3.3). Because the master flow is independent, only the slave flow can be changed for the condition in Eq. (3.3). If the constraint condition in Eq. (3.3) is treated as a super-surface, the slave system should be switched at the super-surface. If the slave and master systems are C^r-continuous and differentiable $(r \geq 1)$ to the super-surface, the slave and master flows will pass through the super-surface instead of staying on the super-surface because of the continuity and differentiation of the slave and master flows. Otherwise, on the super-surface, one obtains $\dot{z} = \varphi^{(1)} = 0$ for all time $t > t_m$ and $\varphi^{(k)} = 0$ for $k = 1, 2, \cdots$. From a theory of discontinuous dynamical systems in Luo [2, 3], at least the slave system possesses discontinuous vector fields to make the slave and master flows stay on the super-surface, which means that the slave and master systems to the constraint can keep the synchronization on the super-surface. Therefore, the constraints can be used as super-surfaces to investigate the synchronization of slave and master systems.

3.1.1 Generalized Synchronization

As discussed in the previous section, if the number of constraints for slave and master systems is over the dimension of the slave state space (i.e., $l > n_s$), the slave system is overconstrained under the constraint conditions by the master system. In other words, if all the constraint conditions are satisfied, the master system should be partially constrained also for $n_s < l \leq n_r + n_s$. Otherwise, the constraint conditions cannot be satisfied for the synchronization of the slave and master systems. The overconstrained synchronization for slave and master systems can be defined from Definition 3.3, i.e.,

Definition 3.5 If $l > n_s$, an $(n_r : n_s; l)$-synchronization of the slave and master systems in Eqs. (3.1) and (3.2) in sense of Eq. (3.4) for time $t \in [t_{m_1}, t_{m_2}]$ is said to be an $(n_r : n_s; l)$-overconstrained synchronization.

To make an overconstrained slave system be synchronized with a master system, the flow of the master system should be controlled by the constraints. Generally speaking, the slave system can be partially controlled by some constraints in Eq. (3.4), and the master system can be partially controlled by the rest constraints in Eq. (3.4) as well. For some time intervals, the slave system can be controlled by the master system under the constraints. With time varying, for some time intervals, the master system can also be controlled by the slave system. For this case, it is very difficult to know which one of two systems is a slave or master system. In fact, it is not necessary to distinguish slave and master systems from two dynamical systems. For the synchronization of two or more systems, Definition 3.2 can be generalized as follows.

Definition 3.6 If a flow $\mathbf{x}^{(s)}(t)$ of a system in Eq. (3.1) with a flow $\mathbf{x}^{(r)}(t)$ of a system in Eq. (3.2) is constrained by a single constraint in Eq. (3.3) for time $t \in [t_{m_1}, t_{m_2}]$, then the two systems are said to be *synchronized* in the sense of Eq. (3.3) for time $t \in [t_{m_1}, t_{m_2}]$. Five special cases are given as follows.

(i) If $t_{m_2} \to \infty$, the two systems are said to be *absolutely* synchronized in the sense of Eq. (3.3) for time $t \in [t_{m_1}, \infty)$.
(ii) If $t_{m_1} \to \infty$, the two systems are said to be *asymptotically* synchronized in the sense of Eq. (3.3).
(iii) For $n_s = n_r = n$, the two *equidimensional* systems are said to be synchronized in the sense of Eq. (3.3) for time $t \in [t_{m_1}, t_{m_2}]$.
(iv) For $n_s = n_r = n$ and $t_{m_2} \to \infty$, the two *equidimensional* systems are said to be *absolutely* synchronized in the sense of Eq. (3.3) for time $t \in [t_{m_1}, \infty)$.
(v) For $n_s = n_r = n$ and $t_{m_1} \to \infty$, the two *equidimensional* systems are said to be *asymptotically* synchronized in the sense of Eq. (3.3).

In an alike fashion, the synchronization of slave and master systems in Definition 3.3 should be generalized for the synchronization of slave and master systems with or without overconstraints.

Definition 3.7 An n_r-dimensional system in Eq. (3.1) with an n_s-dimensional system in Eq. (3.2) is said to be synchronized with l-constraints (or an l-constraint *synchronization*) for time $t \in [t_{m_1}, t_{m_2}]$ if there are l-linearly independent functions $\varphi_j(\mathbf{x}^{(r)}(t), \mathbf{x}^{(s)}(t), t, \boldsymbol{\lambda}_j)$ ($j \in \mathscr{L}$ and $\mathscr{L} = \{1, 2, \cdots, l\}$ with $l < n_r + n_s$) to make two flows $\mathbf{x}^{(r)}(t)$ and $\mathbf{x}^{(s)}(t)$ of the two systems satisfy the constraints in Eq. (3.4) for time $t \in [t_{m_1}, t_{m_2}]$. Five special cases are given as follows:

(i) If $t_{m_2} \to \infty$, the two systems are said to be *absolutely* synchronized with l-constraints (or an absolute, l-constraint synchronization) in the sense of Eq. (3.4) for time $t \in [t_{m_1}, \infty)$.
(ii) If $t_{m_1} \to \infty$, the two systems are said to be *asymptotically* synchronized with l-constraints (*or* an *asymptotic* l-constraint synchronization) in the sense of Eq. (3.4).
(iii) If $n_s = n_r = n$, the two equidimensional systems are said to be synchronized with l-constraints in the sense of Eq. (3.4) for time $t \in [t_{m_1}, t_{m_2}]$.
(iv) If $n_s = n_r = n$ and $t_{m_2} \to \infty$, the two equidimensional systems are said to be *absolutely* synchronized with l-constraints in the sense of Eq. (3.4) for time $t \in [t_{m_1}, \infty)$.
(v) If $n_s = n_r = n$ and $t_{m_1} \to \infty$, the two equidimensional systems are said to be *asymptotically* synchronized with l-constraints in the sense of Eq. (3.4) for time $t \in [t_{m_1}, \infty)$.

From the above definition, the number of constraints in Eq. (3.4) can be greater than the dimension number of state space for any one of the two systems in Eqs. (3.1) and (3.2) (i.e., $l > n_s$ or $l > n_r$). For such case, one cannot control only one of the two systems to make them be synchronized through the constraints. In other words, one must control both of two systems to make the corresponding synchronization occur. Of course, if $l \le n_s$ or $l \le n_r$, one can control only one of two systems to make them be synchronized through the constraints in Eq. (3.4). If the constraint functions $\varphi_j(\mathbf{x}^{(r)}(t), \mathbf{x}^{(s)}(t), t, \boldsymbol{\lambda}_j)$ (for all $j \in \mathscr{L}$) is time independent for $l = n_r + n_s$, Eq. (3.4) will give a set of fixed values of $\mathbf{x}^{(r)*}$ and $\mathbf{x}^{(s)*}$, which are independent of time. The constraints yield the values-fixed, static points in the resultant sate space. To make the two systems in Eqs. (3.1) and (3.2) be synchronized at the static points in phase space, such a synchronization can be called *a static synchronization* of two systems in Eqs. (3.1) and (3.2). For $l > n_s + n_r$, the time-independent constraints in Eq. (3.4) will give the statically overconstrained synchronization, which may not be meaningful for practical problems. Such a case will not be discussed any more. If the constraint functions of $\varphi_j(\mathbf{x}^{(r)}(t), \mathbf{x}^{(s)}(t), t, \boldsymbol{\lambda}_j)$ (for all $j \in \mathscr{L}$) are time dependent for $l = n_r + n_s$, Eq. (3.4) will give a flow of $\mathbf{x}^{(r)*}$ and $\mathbf{x}^{(s)*}$ relative to time. To eliminate time, the constraints in Eq. (3.4) give a one-dimensional flow in the resultant phase space. If the time-dependent constraint functions of $\varphi_j(\mathbf{x}^{(r)}(t), \mathbf{x}^{(s)}(t), t, \boldsymbol{\lambda}_j)$ (for all $j \in \mathscr{L}$) are *of* l-dimensions with $l = n_s + n_r + 1$, Eq. (3.4) will give a set of fixed values of $\mathbf{x}^{(r)*}$ and $\mathbf{x}^{(s)*}$ at a specific time t^* in the resultant phase space, which is an instantaneous fixed point only at time t^*. For this case, it is very difficult for the two systems to be synchronized for such an instantaneous point. Such a case may not be too meaningful, which will not be discussed. Therefore, the following two definitions are given to describe the above-discussed cases.

Definition 3.8 An n_r-dimensional system in Eq. (3.1) with an n_s-dimensional system in Eq. (3.2) is said to be statically synchronized with l-constraints (or *a static synchronization*) for time $t \in [t_{m_1}, t_{m_2}]$ if there are l-linearly independent and *time-independent* functions $\varphi_j(\mathbf{x}^{(r)}(t), \mathbf{x}^{(s)}(t), t, \boldsymbol{\lambda}_j)$ $(j \in \mathscr{L}$ and $\mathscr{L} = \{1, 2, \cdots, l\}$ with $l = n_r + n_s)$ to make two flows $\mathbf{x}^{(r)}(t)$ and $\mathbf{x}^{(s)}(t)$ of the two systems satisfy the constraints in Eq. (3.4) for time $t \in [t_{m_1}, t_{m_2}]$. Two special cases are:

(i) If $t_{m_2} \to \infty$, the two systems are said to be *absolutely* and *statically* synchronized with l-constraints (or an *absolute* and *static* synchronization) in the sense of Eq. (3.4) for time $t \in [t_{m_1}, \infty)$.
(ii) If $t_{m_1} \to \infty$, the two systems are said to be *asymptotically* and *statically* synchronized with l-constraints (*or* an *asymptotic and static* synchronization) in the sense of Eq. (3.4).

Definition 3.9 An n_r-dimensional system in Eq. (3.1) with an n_s-dimensional system in Eq. (3.2) is said to be synchronized with a one-dimensional constraint flow (or a *1D constraint-flow synchronization*) for time $t \in [t_{m_1}, t_{m_2}]$ if there are l linearly independent and *time-dependent* function $\varphi_j(\mathbf{x}^{(r)}(t), \mathbf{x}^{(s)}(t), t, \boldsymbol{\lambda}_j)$ $(j \in \mathscr{L}$ and $\mathscr{L} = \{1, 2, \cdots, l\}$ with $l = n_r + n_s)$ to make two flows $\mathbf{x}^{(r)}(t)$ and $\mathbf{x}^{(s)}(t)$ of the two systems satisfy constraints in Eq. (3.4) for time $t \in [t_{m_1}, t_{m_2}]$. Two special cases are given as follows:

(i) If $t_{m_2} \to \infty$, the two systems are said to be *absolutely* synchronized with a one-dimensional constraint flow (or an *absolute, 1D constraint-flow synchronization*) in the sense of Eq. (3.4) for time $t \in [t_{m_1}, \infty)$.
(ii) If $t_{m_1} \to \infty$, the two systems are said to be *asymptotically* synchronized with a one-dimensional constraint flow (an *asymptotic, 1D constraint-flow synchronization*) in the sense of Eq. (3.4).

3.1.2 Resultant Dynamical Systems

From the theory of discontinuous dynamical systems in Luo [2, 3], the synchronization of two or more dynamical systems with specific constraints can be discussed through a resultant dynamical system. The constraint conditions can be considered as a set of hypersurfaces. If the resultant system to the constraints is discontinuous, the resultant discontinuous dynamical system can be adjusted on both sides of each super-surface for such synchronization. For doing so, a set of new state variables for the resultant discontinuous system will be introduced, and the subdomains and boundaries relative to the constraints will be presented. For synchronization of slave and master systems on the constraint surfaces, only the slave system can be adjusted, and the master system cannot be adjusted. In other words, the slave system can be controlled in order to make it be synchronized with the master system through the constraints. That is, the slave system can be expressed by discontinuous vector fields to all the constraint surfaces for such synchronization, but the master system should keep a continuous vector field to such constraint surfaces. However, for a resultant system formed by two systems with constraints, one can adjust two dynamical systems to make them be synchronized on the constraint conditions in general.

A new vector of state variables of two dynamical systems in Eqs. (3.1) and (3.2) is introduced as

$$\mathbf{y} = (\mathbf{x}^{(r)};\mathbf{x}^{(s)})^{\mathrm{T}} = (x_1^{(r)}, x_2^{(r)}, \cdots, x_{n_r}^{(r)}; x_1^{(s)}, x_2^{(s)}, \cdots, x_{n_s}^{(s)})^{\mathrm{T}} \in \mathscr{R}^{n_r+n_s} \tag{3.16}$$

The notation (•;•) ≡ (•,•) is just for a combined vector of state vectors of two dynamical systems. From the constraint condition in Eq. (3.3), a constraint boundary for the discontinuous description of the synchronization of two dynamical systems in Eqs. (3.1) and (3.2) can be defined, and the corresponding domains separated by such a constraint boundary can be obtained.

Definition 3.10 A constraint boundary in an $(n_r + n_s)$-dimensional phase space for the synchronization of two dynamical systems in Eqs. (3.1) and (3.2) to constraint condition in Eq. (3.3) is defined as

$$\begin{aligned}\partial\Omega_{12} &= \bar{\Omega}_1 \cap \bar{\Omega}_2 \\ &= \left\{ \mathbf{y}^{(0)} \left| \begin{array}{l} \varphi(\mathbf{y}^{(0)}, t, \boldsymbol{\lambda}) \equiv \varphi(\mathbf{x}^{(r;0)}(t), \mathbf{x}^{(s;0)}(t), t, \boldsymbol{\lambda}) = 0, \\ \varphi \text{ is } C^{r_1}\text{-continuous } (r_1 \geq 1) \end{array} \right. \right\} \subset \mathscr{R}^{n_r+n_s-1};\end{aligned} \tag{3.17}$$

and two corresponding domains for a resultant system of two dynamical systems in Eqs. (3.1) and (3.2) are defined as

$$\begin{aligned}\Omega_1 &= \left\{ \mathbf{y}^{(0)} \left| \begin{array}{l} \varphi(\mathbf{y}^{(1)}, t, \boldsymbol{\lambda}) \equiv \varphi(\mathbf{x}^{(r;1)}(t), \mathbf{x}^{(s;1)}(t), t, \boldsymbol{\lambda}) > 0, \\ \varphi \text{ is } C^{r_1}\text{-continuous } (r_1 \geq 1) \end{array} \right. \right\} \subset \mathscr{R}^{n_r+n_s} \\ \Omega_2 &= \left\{ \mathbf{y}^{(0)} \left| \begin{array}{l} \varphi(\mathbf{y}^{(2)}, t, \boldsymbol{\lambda}) \equiv \varphi(\mathbf{x}^{(r;2)}(t), \mathbf{x}^{(s;2)}(t), t, \boldsymbol{\lambda}) < 0, \\ \varphi \text{ is } C^{r_1}\text{-continuous } (r_1 \geq 1) \end{array} \right. \right\} \subset \mathscr{R}^{n_r+n_s};\end{aligned} \tag{3.18}$$

On the two domains, the resultant system of two dynamical systems is discontinuous to the constraint boundary, defined by

$$\dot{\mathbf{y}}^{(\alpha)} = \mathbb{F}^{(\alpha)}(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\pi}^{(\alpha)}) \text{ in } \Omega_\alpha \text{ for } \alpha = 1, 2 \tag{3.19}$$

where $\mathbb{F}^{(\alpha)} = (\mathbf{F}^{(r;\alpha)}, \mathbf{F}^{(s;\alpha)})^{\mathrm{T}}$ and $\boldsymbol{\pi}^{(\alpha)} = (\mathbf{p}^{(r;\alpha)}, \mathbf{p}^{(s;\alpha)})^{\mathrm{T}}$. Suppose there is a vector field $\mathbb{F}^{(0)}(\mathbf{y}^{(0)}, t, \boldsymbol{\lambda})$ on the constraint boundary with $\varphi(\mathbf{y}^{(0)}, t, \boldsymbol{\lambda}) = 0$, and the corresponding dynamical system on such a boundary is expressed by

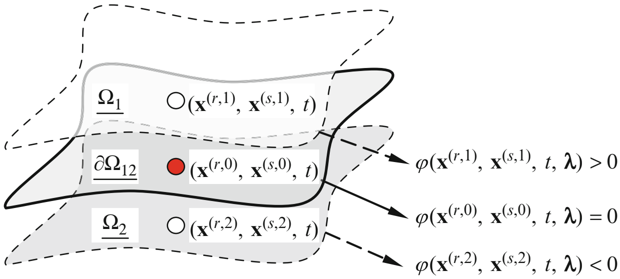

Fig. 3.1 Constraint boundary and domains in $(n_r + n_s)$-dimensional state space

$$\dot{\mathbf{y}}^{(0)} = \mathbb{F}^{(0)}(\mathbf{y}^{(0)}, t, \boldsymbol{\lambda}) \text{ on } \partial\Omega_{12}. \tag{3.20}$$

The domains Ω_α $(\alpha = 1, 2)$ are separated by the constraint boundary $\partial\Omega_{12}$, as shown in Fig. 3.1. For a point $(\mathbf{x}^{(r,1)}, \mathbf{x}^{(s,1)}) \in \Omega_1$ at time t, $\varphi(\mathbf{x}^{(r,1)}, \mathbf{x}^{(s,1)}, t, \boldsymbol{\lambda}) > 0$. For a point $(\mathbf{x}^{(r,2)}, \mathbf{x}^{(s,2)}) \in \Omega_2$ at time t, $\varphi(\mathbf{x}^{(r,2)}, \mathbf{x}^{(s,2)}, t, \boldsymbol{\lambda}) < 0$. However, on the boundary $(\mathbf{x}^{(r,0)}, \mathbf{x}^{(s,0)}) \in \partial\Omega_{12}$ at time t, the constraint condition for synchronization should be satisfied (i.e., $\varphi(\mathbf{x}^{(r,0)}, \mathbf{x}^{(s,0)}, t, \boldsymbol{\lambda}) = 0$). If the constraint condition is time independent, the constraint boundary determined by the constraint condition is invariant. The above definition is extended.

Definition 3.11 The jth-constraint boundary in an $(n_r + n_s)$-dimensional phase space for the synchronization of two dynamical systems in Eqs. (3.1) and (3.2), relative to the jth-constraint of the constraint conditions in Eq. (3.4), is defined as

$$\begin{aligned}
\partial\Omega_{(12,j)} &= \bar{\Omega}_{(1,j)} \cap \bar{\Omega}_{(2,j)} \\
&= \left\{ \mathbf{y}^{(0;j)} \left| \begin{array}{l} \varphi_j(\mathbf{y}^{(0;j)}, t, \boldsymbol{\lambda}_j) \equiv \varphi_j(\mathbf{x}^{(r,0;j)}(t), \mathbf{x}^{(s,0;j)}(t), t, \boldsymbol{\lambda}_j) = 0, \\ \varphi_j \text{ is } C^{r_j}\text{-continuous } (r_j \geq 1) \end{array} \right. \right\} \\
&\subset \mathscr{R}^{n_r+n_s-1};
\end{aligned} \tag{3.21}$$

and two domains pertaining to the jth-boundary for a resultant system of two dynamical systems in Eqs. (3.1) and (3.2) are defined as

$$\begin{aligned}
\Omega_{(1;j)} &= \left\{ \mathbf{y}^{(0)} \left| \begin{array}{l} \varphi_j(\mathbf{y}^{(1;j)}, t, \boldsymbol{\lambda}) \equiv \varphi_j(\mathbf{x}^{(r,1;j)}(t), \mathbf{x}^{(s,1;j)}(t), t, \boldsymbol{\lambda}_j) > 0, \\ \varphi_j \text{ is } C^{r_j}\text{-continuous } (r_j \geq 1) \end{array} \right. \right\} \\
&\subset \mathscr{R}^{n_r+n_s} \\
\Omega_{(2;j)} &= \left\{ \mathbf{y}^{(0)} \left| \begin{array}{l} \varphi_j(\mathbf{y}^{(2;j)}, t, \boldsymbol{\lambda}) \equiv \varphi_j(\mathbf{x}^{(r,2;j)}(t), \mathbf{x}^{(s,2;j)}(t), t, \boldsymbol{\lambda}_j) < 0, \\ \varphi_j \text{ is } C^{r_j}\text{-continuous } (r_j \geq 1) \end{array} \right. \right\} \\
&\subset \mathscr{R}^{n_r+n_s};
\end{aligned} \tag{3.22}$$

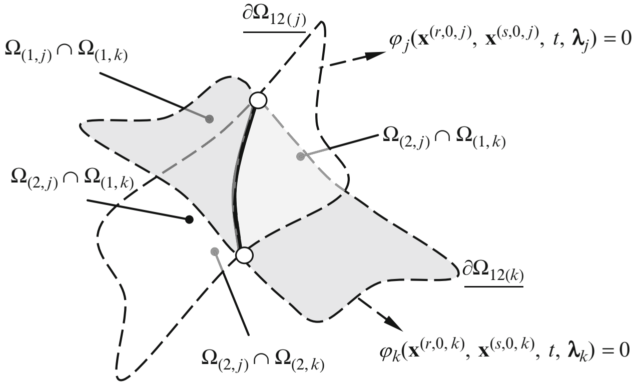

Fig. 3.2 An intersection of two boundaries with $\varphi_j = 0$ and $\varphi_k = 0$ for $j, k \in \mathscr{L}$ and $j \neq k$

On the two domains relative to the jth-constraint boundary, a discontinuous resultant system of two dynamical systems in Eqs. (3.1) and (3.2) with the jth-constraint in Eq. (3.4) is defined by

$$\dot{\mathbf{y}}^{(\alpha_j,j)} = \mathbb{F}^{(\alpha)}(\mathbf{y}^{(\alpha_j,j)}, t, \boldsymbol{\pi}_j^{(\alpha_j)}) \text{ in } \Omega_{(\alpha_j,j)} \quad \text{for } \alpha_j = 1,2 \tag{3.23}$$

where $\mathbb{F}^{(\alpha_j,j)} = (\mathbf{F}^{(r,\alpha_j,j)}; \mathbf{F}^{(s,\alpha_j,j)})^{\mathrm{T}}$ and $\boldsymbol{\pi}_j^{(\alpha_j)} = (\mathbf{p}_j^{(r,\alpha_j)}, \mathbf{p}_j^{(s,\alpha_j)})^{\mathrm{T}}$. Suppose there is a vector field of $\mathbb{F}^{(0,j)}(\mathbf{y}^{(0,j)}, t, \boldsymbol{\lambda}_j)$ on the jth-constraint boundary with $\varphi_j(\mathbf{y}^{(0,j)}, t, \boldsymbol{\lambda}_j) = 0$, and the corresponding dynamical system on the jth-boundary is

$$\dot{\mathbf{y}}^{(0,j)} = \mathbb{F}^{(0;j)}(\mathbf{y}^{(0,j)}, t, \boldsymbol{\lambda}_j) \text{ in } \Omega_{(12,j)} \quad \text{for } \alpha_j = 1,2 \tag{3.24}$$

Since l-constraint conditions are linearly independent, any two boundaries are intersected each other. Consider two constraint boundaries of $\partial\Omega_{12(j)}$ and $\partial\Omega_{12(k)}$ for synchronization. From Luo [4], the intersection edge of the two constraint boundaries is given by

$$\partial\Omega_{(12,jk)} = \partial\Omega_{(12,j)} \cap \partial\Omega_{(12,k)} \subset \mathscr{R}^{n_r+n_s-2} \tag{3.25}$$

and the corresponding domain in phase space is separated into four subdomains

$$\Omega_{(\alpha_j\alpha_k,jk)} = \Omega_{(\alpha_j,j)} \cap \Omega_{(\alpha_k,k)} \subset \mathscr{R}^{n_r+n_s} \quad \text{for } j,k \in \mathscr{L} \text{ and } \alpha_j,\alpha_k = 1,2. \tag{3.26}$$

Such a partition of the domain in state space for a resultant system of two dynamical systems is sketched in Fig. 3.2. The intersection of the two constraint boundaries in state space for a resultant system of two dynamical systems is depicted by an $(n_r + n_s - 2)$-manifold, depicted by a dark curve. For the l-linearly independent constraints, the state space partition can be completed via such l-linearly independent constraint boundaries. Based on the l-constraint conditions, the corresponding intersection of boundaries is

$$\partial\Omega_{(12,\mathbf{J})} = \cap_{j=1}^{l}\partial\Omega_{(12,j)} \subset \mathscr{R}^{n_r+n_s-l}. \tag{3.27}$$

which gives an $(n_r + n_s - l)$-dimensional edge manifold. Consider the synchronization of the slave and master systems for discussion. If $n = l$, the intersection manifold of the constraints is an n_s-dimensional state space. Thus, the slave system can be completely controlled through the n_s-constraints to be synchronized with the master system. From the l-constraint conditions in Eq. (3.4), the domain in $(n_r + n_s)$ -dimensional state space is partitioned into many subdomains for the resultant system of two dynamical systems, i.e.,

$$\Omega_{\boldsymbol{\alpha}} = \Omega_{(\alpha_1\alpha_2\cdots\alpha_l)} = \cap_{j=1}^{l}\Omega_{(\alpha_j,j)} \subset \mathscr{R}^{n_r+n_s} \text{ for } \alpha_j = 1,2 \text{ and } j \in \mathscr{L}. \tag{3.28}$$

The total domain $\mho = \cup_{j=1}^{l}\cup_{\alpha_j=1}^{2}\left(\cap_{j=1}^{l}\Omega_{(\alpha_j,j)}\right) \subset \mathscr{R}^{n_r+n_s}$ is a union of all the subdomains. From the foregoing description of a resultant dynamical system, the synchronization of two systems under constraints can be investigated through such a resultant dynamical system with the constraint boundaries as in Luo [2, 3]. The constraint boundaries can be either of one side or of two sides. If the resultant system for the synchronization of two systems can be defined in one of the two subdomains only, such a constraint boundary is called one-side boundary. Otherwise, the constraint boundary is called two-side constraint boundary. If a flow of the resultant system can approach to a constraint flow on the constraint boundaries as $t \to \infty$, for such a case, the synchronization of two systems to the constraint boundaries is asymptotic.

3.2 Synchronization with a Single Constraint

In this section, the synchronicity of two systems to a single constraint will be presented, and the corresponding conditions for such synchronicity will be discussed.

3.2.1 Synchronicity

Before discussing the synchronicity of two dynamical systems to the constraint boundary, the neighborhood of the constraint boundary should be introduced through a typical point on such a constraint boundary for time t_m. For any small $\varepsilon > 0$, the neighborhood of a constraint boundary is defined as follows.

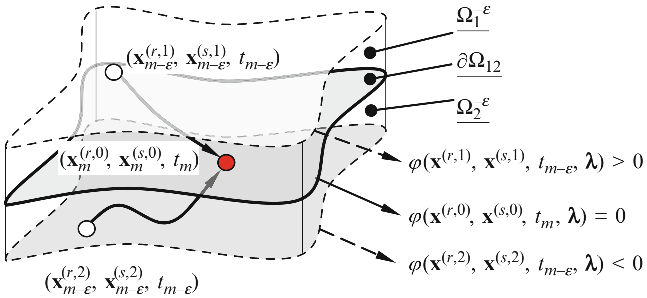

Fig. 3.3 A neighborhood of the constraint boundary and the attractivity of a resultant flow to the constraint boundary in $(n_r + n_s)$-dimensional state space

Definition 3.12 For $\mathbf{y}_{m\pm}^{(\alpha)} \in \Omega_\alpha$ ($\alpha \in \{1,2\}$) and $\mathbf{y}_m^{(0)} \in \partial\Omega_{12}$ at time $t_m, \mathbf{y}_{m\pm}^{(\alpha)} = \mathbf{y}_m^{(0)}$. For any small $\varepsilon > 0$, there is a time interval $[t_{m-\varepsilon}, t_m)$ or $(t_m, t_{m+\varepsilon}]$. The ε-neighborhood of the constraint boundary $\partial\Omega_{12}$ is defined as

$$\begin{aligned}\Omega_\alpha^{-\varepsilon} &= \left\{\mathbf{y}^{(\alpha)} \middle| \|\mathbf{y}^{(\alpha)}(t) - \mathbf{y}_m^{(0)}\| \le \delta, \delta > 0, t \in [t_{m-\varepsilon}, t_m)\right\},\\ \Omega_\alpha^{+\varepsilon} &= \left\{\mathbf{y}^{(\alpha)} \middle| \|\mathbf{y}^{(\alpha)}(t) - \mathbf{y}_m^{(0)}\| \le \delta, \delta > 0, t \in (t_m, t_{m+\varepsilon}]\right\}.\end{aligned} \tag{3.29}$$

For a point $\mathbf{y}_m^{(0)} = (\mathbf{x}_m^{(r,0)}, \mathbf{x}_m^{(s,0)})^{\mathrm{T}} \in \partial\Omega_{12}$ at time t_m, a surface of the constraint boundary $\partial\Omega_{12}$ at the instantaneous time t_m is governed by $\varphi(\mathbf{x}^{(r,0)}, \mathbf{x}^{(s,0)}, t_m, \boldsymbol{\lambda}) = \varphi(\mathbf{x}_m^{(r,0)}, \mathbf{x}_m^{(s,0)}, t_m, \boldsymbol{\lambda}) = 0$. If the constraint function φ is time independent, such a constraint surface for the synchronization of two dynamical systems is invariant with respect to time. Otherwise, this constraint surface changes with the instantaneous time t_m. In addition to the constraint surface, two boundaries of domain $\Omega_\alpha^{-\varepsilon}$ ($\alpha = 1,2$) are determined by $\varphi(\mathbf{x}^{(r,\alpha)}, \mathbf{x}^{(s,\alpha)}, t_{m-\varepsilon}, \boldsymbol{\lambda}) = \varphi(\mathbf{x}_{m-\varepsilon}^{(r,\alpha)}, \mathbf{x}_{m-\varepsilon}^{(s,\alpha)}, t_{m-\varepsilon}, \boldsymbol{\lambda}) = \text{const}$, as shown in Fig. 3.3. In the ε-neighborhood of a constraint boundary, if the resultant system of two dynamical systems is attractive to such a constraint boundary, any flows in the two ε-domains will approach the constraint boundary. Further, the synchronicity of two dynamical systems to the constraint boundary can be discussed. In other words, the attractivity of the resultant system to the constraint boundary requires that any flow in the two ε-domains of Ω_α ($\alpha = 1,2$) approach the constraint boundary $\partial\Omega_{12}$ as $t \to t_m$. From Luo [2, 3], the synchronization of two dynamical systems to the constraint needs that any flows of the resultant system in the two ε-domains of Ω_α ($\alpha = 1,2$) are attractive to the boundary.

Definition 3.13 Consider two dynamical systems in Eqs. (3.1) and (3.2) with a constraint in Eq. (3.3). For $\mathbf{y}_{m\pm}^{(\alpha)} \in \Omega_\alpha$ ($\alpha \in \{1,2\}$) and $\mathbf{y}_m^{(0)} \in \partial\Omega_{12}$ at time t_m, $\mathbf{y}_{m\pm}^{(\alpha)} = \mathbf{y}_m^{(0)}$. For any small $\varepsilon > 0$, there is a time interval $[t_{m-\varepsilon}, t_m)$. The two systems in Eqs. (3.1) and (3.2) to constraint in Eq. (3.3) are called to be *synchronized* for time $t_m \in [t_{m_1}, t_{m_2}]$ if

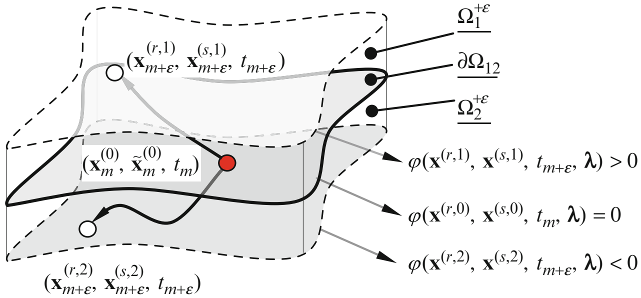

Fig. 3.4 The repulsion of a resultant flow to the constraint boundary in $(n_r + n_s)$-dimensional state space

$$\begin{aligned}&\varphi(\mathbf{y}_{m-}^{(\alpha)}, t_{m-}, \boldsymbol{\lambda}) = \varphi(\mathbf{y}_m^{(0)}, t_m, \boldsymbol{\lambda}) = 0;\\&(-1)^{\alpha}[\varphi(\mathbf{y}_{m-\varepsilon}^{(\alpha)}, t_{m-\varepsilon}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}_{m-}^{(\alpha)}, t_{m-}, \boldsymbol{\lambda})] < 0 \quad \text{for } \alpha = 1, 2.\end{aligned} \tag{3.30}$$

In addition to the attractivity of a flow of the resultant system to the constraint boundary, the repulsion of a flow of the resultant system to the constraint boundary can be defined. Because of such a repulsion, any flows of the resultant system in the two ε-domains of Ω_α ($\alpha = 1, 2$) can never approach the constraint boundary. In other words, two dynamical systems in Eqs. (3.1) and (3.2) cannot make the constraint condition in Eq. (3.3) be satisfied. Thus, the repulsion of a flow of the resultant system to the constraint boundary should be introduced. Such a repulsion phenomenon is sketched in Fig. 3.4. The constraint boundary $\partial\Omega_{12}$ is governed by $\varphi(\mathbf{x}^{(r,0)}, \mathbf{x}^{(s,0)}, t_m, \boldsymbol{\lambda}) = 0$. The boundaries of the ε-neighborhood of the constraint boundary are obtained by $\varphi(\mathbf{x}^{(r,\alpha)}, \mathbf{x}^{(s,\alpha)}, t_{m+\varepsilon}, \boldsymbol{\lambda}) = \varphi(\mathbf{x}_{m+\varepsilon}^{(r,\alpha)}, \mathbf{x}_{m+\varepsilon}^{(s,\alpha)}, t_{m+\varepsilon}, \boldsymbol{\lambda}) = \text{const}$. Two flows of the resultant system on both sides of the constraint boundary $\partial\Omega_{12}$ move away in two domains Ω_α ($\alpha = 1, 2$), which means that no any flows of the resultant system can arrive to the constraint boundary. So the synchronization of two dynamical systems in Eqs. (3.1) and (3.2) to the constraint in Eq. (3.3) cannot be achieved. Such a repulsion of a resultant system to the constraint boundary gives the desynchronization of two dynamical systems to the constraint in Eq. (3.3). The desynchronization of two systems to a constraint is defined.

Definition 3.14 Consider two systems in Eqs. (3.1) and (3.2) with constraint in Eq. (3.3). For $\mathbf{y}_{m\pm}^{(\alpha)} \in \Omega_\alpha$ ($\alpha \in \{1, 2\}$) and $\mathbf{y}_m^{(0)} \in \partial\Omega_{12}$ at time t_m, $\mathbf{y}_{m\pm}^{(\alpha)} = \mathbf{y}_m^{(0)}$. For any small $\varepsilon > 0$, there is a time interval $(t_m, t_{m+\varepsilon}]$. The two dynamical systems in Eqs. (3.1) and (3.2) to constraint in Eq. (3.3) are said to *be repelled* (*or desynchronized*) for $t_m \in [t_{m_1}, t_{m_2}]$ if

$$\begin{aligned}&\varphi(\mathbf{y}_{m+}^{(\alpha)}, t_{m+}, \boldsymbol{\lambda}) = \varphi(\mathbf{y}_m^{(0)}, t_m, \boldsymbol{\lambda}) = 0;\\&(-1)^{\alpha}[\varphi(\mathbf{y}_{m+\varepsilon}^{(\alpha)}, t_{m+\varepsilon}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}_{m+}^{(\alpha)}, t_{m+}, \boldsymbol{\lambda})] < 0 \quad \text{for } \alpha = 1, 2.\end{aligned} \tag{3.31}$$

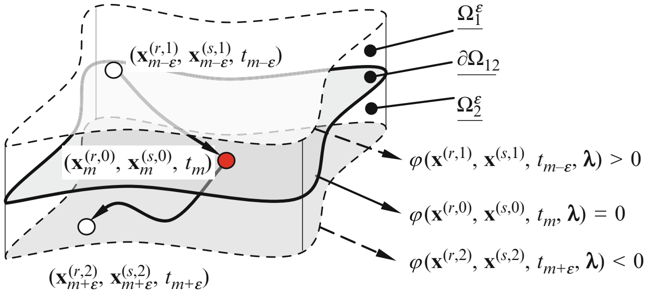

Fig. 3.5 A penetration of a resultant flow to the constraint boundary in $(n_r + n_s)$-dimensional state space

From the theory of discontinuous dynamical systems in Luo [2, 3], a resultant system of two dynamical systems in Eqs. (3.1) and (3.2) may pass through the constraint boundary from a domain to another. For this case, the penetration synchronicity of two dynamical systems can occur, as sketched in Fig. 3.5. Such synchronization can be called *an instantaneous synchronization*. A flow of a resultant system to the constraint boundary for time $t < t_m$ and $t > t_m$ lies in the two domains Ω_1 and Ω_2. In sense of Eq. (3.3), a definition of such penetration synchronicity is given as follows.

Definition 3.15 Consider two dynamical systems in Eqs. (3.1) and (3.2) with constraint in Eq. (3.3). For $\mathbf{y}_{m\pm}^{(\alpha)} \in \Omega_\alpha$ $(\alpha \in \{1,2\})$ and $\mathbf{y}_m^{(0)} \in \partial\Omega_{12}$ at time t_m, $\mathbf{y}_{m\pm}^{(\alpha)} = \mathbf{y}_m^{(0)}$. For any small $\varepsilon > 0$, there is a time interval $[t_{m-\varepsilon}, t_{m+\varepsilon}]$. A resultant flow of two dynamical systems in Eqs. (3.1) and (3.2) is said to be *penetrated* to the constraint boundary $\partial\Omega_{\alpha\beta}$ from Ω_α to Ω_β at time t_m if for $\alpha, \beta \in \{1,2\}$ and $\alpha \neq \beta$

$$\left.\begin{aligned} &\varphi(\mathbf{y}_{m-}^{(\alpha)}, t_{m-}, \boldsymbol{\lambda}) = \varphi(\mathbf{y}_{m+}^{(\beta)}, t_{m+}, \boldsymbol{\lambda}) = \varphi(\mathbf{y}_m^{(0)}, t_m, \boldsymbol{\lambda}) = 0;\\ &(-1)^\alpha[\varphi(\mathbf{y}_{m-\varepsilon}^{(\alpha)}, t_{m-\varepsilon}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}_{m-}^{(\alpha)}, t_{m-}, \boldsymbol{\lambda})] < 0;\\ &(-1)^\beta[\varphi(\mathbf{y}_{m+\varepsilon}^{(\beta)}, t_{m+\varepsilon}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}_{m+}^{(\beta)}, t_{m+}, \boldsymbol{\lambda})] < 0. \end{aligned}\right\} \tag{3.32}$$

In Definition 3.15, the incoming flow with "$-$" and outcome flow with "$+$" to the boundary are prescribed. From the foregoing definition, a penetration flow of the resultant system of two dynamical systems to the constraint boundary can be considered to be formed by the semi-synchronization and semi-desynchronization. Such a penetration flow of the resultant system to the constraint boundary can also be called *an instantaneous synchronization* of two dynamical systems in Eqs (3.1) and (3.2) to constraint in Eq. (3.3). Such an instantaneous synchronization will disappear because the semi-desynchronization exists. From the definition of a penetration flow, a flow of the resultant system in domain Ω_α approaches the constraint boundary. However, in domain Ω_β, such a flow will leave from the constraint boundary. To investigate the

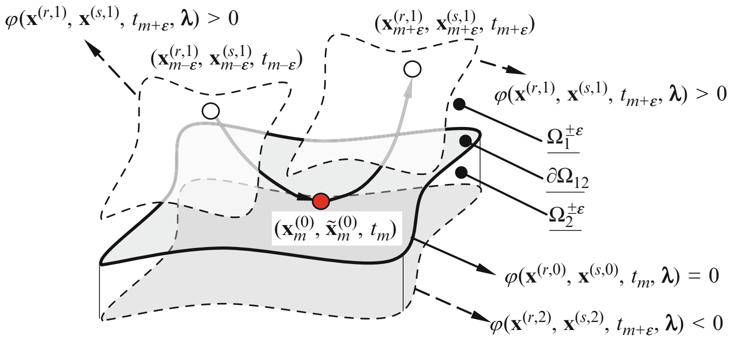

Fig. 3.6 Tangential synchronization to the constraint in an $(n_r + n_s)$-dimensional state space

relations among three types of synchronicity of two dynamical systems to the constraint in Eq. (3.3), the switchability of the synchronization, desynchronization, and penetration is very important, which can be discussed through the singularity of the resultant system to the constraint boundary.

3.2.2 Singularity to Constraint

From a theory of discontinuous dynamical systems in Luo [2–4], a flow of a resultant system of two dynamical systems may be tangential to the constraint boundary governed by the constraint condition in Eq. (3.3). For this case, the synchronicity of two dynamical systems to the constraint occurs only at one point and then returns back to the same domain. Such an *instantaneous* synchronization is different from a penetration flow of the resultant system to the constraint boundary. The tangential synchronization of two dynamical systems to the constraint is sketched in Fig. 3.6. In domain Ω_1, the tangential synchronization of the two systems to the constraint boundary $\partial\Omega_{12}$ is presented. The two boundaries at time $t_{m-\varepsilon}$ and $t_{m+\varepsilon}$ are given by the two different surfaces. For such synchronicity, the following definition is given.

Definition 3.16 Consider two dynamical systems in Eqs. (3.1) and (3.2) with constraint in Eq. (3.3). For $\mathbf{y}^{(\alpha)}_{m\pm} \in \Omega_\alpha$ ($\alpha \in \{1,2\}$) and $\mathbf{y}^{(0)}_{m} \in \partial\Omega_{12}$ at time t_m, $\mathbf{y}^{(\alpha)}_{m\pm} = \mathbf{y}^{(0)}_{m}$. For any small $\varepsilon > 0$, there is a time interval $[t_{m-\varepsilon}, t_{m+\varepsilon}]$. At $\mathbf{y}^{(\alpha)} \in \Omega^{\pm\varepsilon}_\alpha$ for $t \in [t_{m-\varepsilon}, t_{m+\varepsilon}]$, the function $\varphi(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\lambda})$ is C^{r_α}-continuous ($r_\alpha \geq 2$) and $|\varphi^{(r_\alpha+1)}(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\lambda})| < \infty$. A flow of a resultant system of two dynamical systems in Eqs. (3.1) and (3.2) is said to be *tangential* (or grazing) to the constraint boundary at time t_m if for $\alpha \in \{1,2\}$

$$\begin{aligned}
&\varphi(\mathbf{y}_{m\pm}^{(\alpha)}, t_{m\pm}, \boldsymbol{\lambda}) = \varphi(\mathbf{y}_{m}^{(0)}, t_{m}, \boldsymbol{\lambda}) = 0;\\
&\varphi^{(1)}(\mathbf{y}_{m\pm}^{(\alpha)}, t_{m\pm}, \boldsymbol{\lambda}) = 0;\\
&(-1)^{\alpha}[\varphi(\mathbf{y}_{m\pm\varepsilon}^{(\alpha)}, t_{m\pm\varepsilon}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}_{m\pm}^{(\alpha)}, t_{m\pm}, \boldsymbol{\lambda})] < 0.
\end{aligned} \tag{3.33}$$

In Definition 3.16, the incoming flow with " $-$ " and outcome flow with " $+$ " to the boundary are prescribed. Such a tangency of a resultant flow to the constraint boundary will cause the synchronicity to be changed. The onset and vanishing singularity for synchronizations can be discussed, and the corresponding definition is given as follows.

Definition 3.17 Consider two dynamical systems in Eqs. (3.1) and (3.2) with constraint in Eq. (3.3). For $\mathbf{y}_m^{(\alpha)} \in \Omega_\alpha$ $(\alpha \in \{1,2\})$ and $\mathbf{y}_m^{(0)} \in \partial\Omega_{12}$ at time t_m, $\mathbf{y}_m^{(\alpha)} = \mathbf{y}_m^{(0)}$. For any small $\varepsilon > 0$, there is a time interval $[t_{m-\varepsilon}, t_{m+\varepsilon}]$. At $\mathbf{y}^{(\alpha)} \in \Omega_\alpha^{\pm\varepsilon}$ for time $t \in [t_{m-\varepsilon}, t_{m+\varepsilon}]$, the constraint function $\varphi(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\lambda})$ is C^{r_α}-continuous ($r_\alpha \geq 2$) and $|\varphi^{(r_\alpha+1)}(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\lambda})| < \infty$

(i) The *synchronization* of two dynamical systems in Eqs. (3.1) and (3.2) with constraint in Eq. (3.3) is called to be *vanishing* to form a penetration from domain Ω_α to Ω_β at the constraint boundary at time t_m if for $\alpha, \beta \in \{1,2\}$ and $\alpha \neq \beta$

$$\begin{aligned}
&\varphi(\mathbf{y}_{m-}^{(\alpha)}, t_{m-}, \boldsymbol{\lambda}) = \varphi(\mathbf{y}_{m\mp}^{(\beta)}, t_{m\mp}, \boldsymbol{\lambda}) = \varphi(\mathbf{y}_{m}^{(0)}, t_{m}, \boldsymbol{\lambda}) = 0;\\
&\varphi^{(1)}(\mathbf{y}_{m-}^{(\alpha)}, t_{m-}, \boldsymbol{\lambda}) \neq 0, \varphi^{(1)}(\mathbf{y}_{m\mp}^{(\beta)}, t_{m\mp}, \boldsymbol{\lambda}) = 0;\\
&(-1)^{\alpha}[\varphi(\mathbf{y}_{m-\varepsilon}^{(\alpha)}, t_{m-\varepsilon}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}_{m-}^{(\alpha)}, t_{m-}, \boldsymbol{\lambda})] < 0;\\
&(-1)^{\beta}[\varphi(\mathbf{y}_{m\mp\varepsilon}^{(\beta)}, t_{m\mp\varepsilon}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}_{m\mp}^{(\beta)}, t_{m\mp}, \boldsymbol{\lambda})] < 0.
\end{aligned} \tag{3.34}$$

(ii) The synchronization of two dynamical systems in Eqs. (3.1) and (3.2) with constraint in Eq. (3.3) is called to be *onset* from a penetration from domain Ω_α to Ω_β at the constraint boundary at time t_m if for $\alpha, \beta \in \{1,2\}$ and $\alpha \neq \beta$

$$\begin{aligned}
&\varphi(\mathbf{y}_{m-}^{(\alpha)}, t_{m-}, \boldsymbol{\lambda}) = \varphi(\mathbf{y}_{m\pm}^{(\beta)}, t_{m\pm}, \boldsymbol{\lambda}) = \varphi(\mathbf{y}_{m}^{(0)}, t_{m}, \boldsymbol{\lambda}) = 0;\\
&\varphi^{(1)}(\mathbf{y}_{m-}^{(\alpha)}, t_{m-}, \boldsymbol{\lambda}) \neq 0, \varphi^{(1)}(\mathbf{y}_{m\pm}^{(\beta)}, t_{m\pm}, \boldsymbol{\lambda}) = 0;\\
&(-1)^{\alpha}[\varphi(\mathbf{y}_{m-\varepsilon}^{(\alpha)}, t_{m-\varepsilon}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}_{m-}^{(\alpha)}, t_{m-}, \boldsymbol{\lambda})] < 0;\\
&(-1)^{\beta}[\varphi(\mathbf{y}_{m\pm\varepsilon}^{(\beta)}, t_{m\pm\varepsilon}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}_{m\pm}^{(\beta)}, t_{m\pm}, \boldsymbol{\lambda})] < 0,
\end{aligned} \tag{3.35}$$

In Eq. (3.34), the notation " $\mp$ " represents the synchronization first with " $-$ " and the penetration secondly with "+". This condition is called either the *vanishing* condition of synchronization to form a new penetration or the *onset* condition of *penetration* from the synchronization at the boundary of constraint in Eq. (3.3). However, in Eq. (3.35), the notation " $\pm$ " represents the penetration first with "+"

and the synchronization secondly with " $-$." This condition is called the *onset* condition of *synchronization* from a state of penetration to the boundary, which can also be called the *vanishing* condition of *penetration* to form a synchronization at the constraint boundary at time t_m. The switching conditions between the synchronization and desynchronization are presented as follows.

Definition 3.18 Consider two dynamical systems in Eqs. (3.1) and (3.2) with constraint in Eq. (3.3). For $\mathbf{y}_{m\pm}^{(\alpha)} \in \Omega_\alpha\,(\alpha \in \{1,2\})$ and $\mathbf{y}_m^{(0)} \in \partial\Omega_{12}$ at time t_m, $\mathbf{y}_{m\pm}^{(\alpha)} = \mathbf{y}_m^{(0)}$. For any small $\varepsilon > 0$, there is a time interval $[t_{m-\varepsilon}, t_{m+\varepsilon}]$. At $\mathbf{y}^{(\alpha)} \in \Omega_\alpha^{\pm\varepsilon}$ for time $t \in [t_{m-\varepsilon}, t_{m+\varepsilon}]$, the constraint function $\varphi(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\lambda})$ is C^{r_α}-continuous ($r_\alpha \geq 2$) and $|\varphi^{(r_\alpha+1)}(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\lambda})| < \infty$.

(i) The *synchronization of* two dynamical systems in Eqs. (3.1) and (3.2) to constraint in Eq. (3.3) is called to be *onset* from a desynchronization at the constraint boundary at time t_m if for $\alpha = 1, 2$

$$\begin{aligned}
&\varphi(\mathbf{y}_{m\pm}^{(\alpha)}, t_{m\pm}, \boldsymbol{\lambda}) = \varphi(\mathbf{y}_m^{(0)}, t_m, \boldsymbol{\lambda}) = 0;\\
&\varphi^{(1)}(\mathbf{y}_{m\pm}^{(\alpha)}, t_{m\pm}, \boldsymbol{\lambda}) = 0;\\
&(-1)^\alpha[\varphi(\mathbf{y}_{m\pm\varepsilon}^{(\alpha)}, t_{m\pm\varepsilon}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}_{m\pm}^{(\alpha)}, t_{m\pm}, \boldsymbol{\lambda})] < 0.
\end{aligned} \tag{3.36}$$

(ii) The synchronization of two dynamical systems in Eqs. (3.1) and (3.2) to constraint in Eq. (3.3) is called to be *vanishing* to form a desynchronization at the constraint boundary at time t_m if for $\alpha = 1, 2$

$$\begin{aligned}
&\varphi(\mathbf{y}_{m\mp}^{(\alpha)}, t_{m\mp}, \boldsymbol{\lambda}) = \varphi(\mathbf{y}_m^{(0)}, t_m, \boldsymbol{\lambda}) = 0;\\
&\varphi^{(1)}(\mathbf{y}_{m\mp}^{(\alpha)}, t_{m\mp}, \boldsymbol{\lambda}) = 0;\\
&(-1)^\alpha[\varphi(\mathbf{y}_{m\mp\varepsilon}^{(\alpha)}, t_{m\mp\varepsilon}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}_{m\mp}^{(\alpha)}, t_{m\mp}, \boldsymbol{\lambda})] < 0.
\end{aligned} \tag{3.37}$$

In Eq. (3.36), the notation " $\pm$ " represents the desynchronization first with "+" and the synchronization with " $-$ " second. This condition is called either *the onset condition of synchronization* from the desynchronization on the boundary or *the vanishing condition of desynchronization* to form a new synchronization on the boundary. In Eq. (3.37), the notation " $\mp$ " represents the synchronization first with " $-$ " and the desynchronization second with "+". This condition is called *the vanishing condition of synchronization* to form a new desynchronization, which can also be called *the onset condition of desynchronization* from the synchronization. Similarly, the onset and vanishing conditions of the desynchronization from the penetration can be discussed as for the synchronization. The following definition will give the onset and vanishing conditions of desynchronization.

Definition 3.19 Consider two dynamical systems in Eqs. (3.1) and (3.2) with constraint in Eq. (3.3). For $\mathbf{y}_{m\pm}^{(\alpha)} \in \Omega_\alpha$ $(\alpha \in \{1,2\})$ and $\mathbf{y}_m^{(0)} \in \partial\Omega_{12}$ at time t_m,

$\mathbf{y}_{m\pm}^{(\alpha)} = \mathbf{y}_m^{(0)}$. For any small $\varepsilon > 0$, there is a time interval $[t_{m-\varepsilon}, t_{m+\varepsilon}]$. At $\mathbf{y}^{(\alpha)} \in \Omega_\alpha^{\pm\varepsilon}$ for time $t \in [t_{m-\varepsilon}, t_{m+\varepsilon}]$, the constraint function $\varphi(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\lambda})$ is C^{r_α}-continuous $(r_\alpha \geq 2)$ and $|\varphi^{(r_\alpha+1)}(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\lambda})| < \infty$.

(i) The desynchronization of two dynamical systems in Eqs. (3.1) and (3.2) to constraint in Eq. (3.3) is called to be *vanishing* to form a penetration from Ω_α to Ω_β at the constraint boundary at time t_m if for $\alpha, \beta \in \{1, 2\}$ and $\alpha \neq \beta$

$$\begin{aligned}
&\varphi(\mathbf{y}_{m\pm}^{(\alpha)}, t_{m\pm}, \boldsymbol{\lambda}) = \varphi(\mathbf{y}_{m+}^{(\beta)}, t_{m+}, \boldsymbol{\lambda}) = \varphi(\mathbf{y}_m^{(0)}, t_m, \boldsymbol{\lambda}) = 0;\\
&\varphi^{(1)}(\mathbf{y}_{m\pm}^{(\alpha)}, t_{m\pm}, \boldsymbol{\lambda}) = 0, \varphi^{(1)}(\mathbf{y}_{m+}^{(\beta)}, t_{m+}, \boldsymbol{\lambda}) \neq 0;\\
&(-1)^\alpha[\varphi(\mathbf{y}_{m\pm\varepsilon}^{(\alpha)}, t_{m\pm\varepsilon}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}_{m\pm}^{(\alpha)}, t_{m\pm}, \boldsymbol{\lambda})] < 0,\\
&(-1)^\beta[\varphi(\mathbf{y}_{m+\varepsilon}^{(\beta)}, t_{m+\varepsilon}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}_{m+}^{(\beta)}, t_{m+}, \boldsymbol{\lambda})] < 0.
\end{aligned} \tag{3.38}$$

(ii) The desynchronization of two dynamical systems in Eqs. (3.1) and (3.2) to constraint in Eq. (3.3) is called to be *onset* from a penetration from Ω_α to Ω_β at the constraint boundary at time t_m if for $\alpha, \beta \in \{1, 2\}$ and $\alpha \neq \beta$

$$\begin{aligned}
&\varphi(\mathbf{y}_{m\mp}^{(\alpha)}, t_{m\mp}, \boldsymbol{\lambda}) = \varphi(\mathbf{y}_{m+}^{(\beta)}, t_{m+}, \boldsymbol{\lambda}) = \varphi(\mathbf{y}_m^{(0)}, t_m, \boldsymbol{\lambda}) = 0;\\
&\varphi^{(1)}(\mathbf{y}_{m\mp}^{(\alpha)}, t_{m\mp}, \boldsymbol{\lambda}) = 0,\ \varphi^{(1)}(\mathbf{y}_{m+}^{(\beta)}, t_{m+}, \boldsymbol{\lambda}) \neq 0;\\
&(-1)^\alpha[\varphi(\mathbf{y}_{m\mp\varepsilon}^{(\alpha)}, t_{m\mp\varepsilon}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}_{m\mp}^{(\alpha)}, t_{m\mp}, \boldsymbol{\lambda})] < 0;\\
&(-1)^\beta[\varphi(\mathbf{y}_{m+\varepsilon}^{(\beta)}, t_{m+\varepsilon}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}_{m+}^{(\beta)}, t_{m+}, \boldsymbol{\lambda})] > 0.
\end{aligned} \tag{3.39}$$

Notice that in Eq. (3.38), the notation "$\pm$" represents the desynchronization first with "+" and the penetration second with "$-$". This condition is called the *vanishing* condition of *desynchronization* to form a new penetration on the boundary and can also be called the *onset* condition of penetration from a synchronization state. However, in Eq. (3.39), the notation "$\mp$" represents the penetration first with "$-$" and the synchronization second with "$+$". This condition is called *the onset condition of desynchronization* from a penetration and also can be called *the vanishing condition of the penetration* to form a desynchronization state. From the previous three definitions, the switching between synchronization and penetration, between desynchronization and penetration, and between desynchronization and synchronization were presented. However, another switching between two penetrations should be discussed.

Definition 3.20 Consider two dynamical systems in Eqs. (3.1) and (3.2) with constraint in Eq. (3.3). For $\mathbf{y}_{m\pm}^{(\alpha)} \in \Omega_\alpha (\alpha \in \{1, 2\})$ and $\mathbf{y}_m^{(0)} \in \partial\Omega_{12}$ at time t_m, $\mathbf{y}_{m\pm}^{(\alpha)} = \mathbf{y}_m^{(0)}$. For any small $\varepsilon > 0$, there is a time interval $[t_{m-\varepsilon}, t_{m+\varepsilon}]$. At $\mathbf{y}^{(\alpha)} \in \Omega_\alpha^{\pm\varepsilon}$ for time $t \in [t_{m-\varepsilon}, t_{m+\varepsilon}]$, the constraint function $\varphi(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\lambda})$ is C^{r_α}-continuous $(r_\alpha \geq 2)$ and $|\varphi^{(r_\alpha+1)}(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\lambda})| < \infty$. The penetration of the two dynamical systems in

Eqs. (3.1) and (3.2) to constraint in Eq. (3.3) is called to be *switched* at the constraint boundary at time t_m if for $\alpha, \beta \in \{1,2\}$

$$\begin{aligned}
&\varphi(\mathbf{y}_{m\mp}^{(\alpha)}, t_{m\mp}, \boldsymbol{\lambda}) = \varphi(\mathbf{y}_{m\pm}^{(\beta)}, t_{m\pm}, \boldsymbol{\lambda}) = \varphi(\mathbf{y}_{m}^{(0)}, t_{m}, \boldsymbol{\lambda}) = 0;\\
&\varphi^{(1)}(\mathbf{y}_{m\mp}^{(\alpha)}, t_{m\mp}, \boldsymbol{\lambda}) = \varphi^{(1)}(\mathbf{y}_{m\pm}^{(\beta)}, t_{m\pm}, \boldsymbol{\lambda}) = 0;\\
&(-1)^{\alpha}[\varphi(\mathbf{y}_{m\mp\varepsilon}^{(\alpha)}, t_{m\mp\varepsilon}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}_{m\mp}^{(\alpha)}, t_{m\mp}, \boldsymbol{\lambda})] < 0,\\
&(-1)^{\beta}[\varphi(\mathbf{y}_{m\pm\varepsilon}^{(\beta)}, t_{m\pm\varepsilon}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}_{m\pm}^{(\beta)}, t_{m\pm}, \boldsymbol{\lambda})] < 0.
\end{aligned} \tag{3.40}$$

Based on the definitions of the tangential (or grazing) and switching singularity, there is a critical parameter $\boldsymbol{\lambda}_{cr}$ from which $\partial\varphi(\mathbf{y}_{m\pm}^{(\alpha)}, t_{m\pm}, \boldsymbol{\lambda})/\partial\boldsymbol{\lambda}|_{\boldsymbol{\lambda}_{cr}} \neq 0$, such a singularity is called the corresponding bifurcation at $\boldsymbol{\lambda}_{cr}$for parameter $\boldsymbol{\lambda}$.

3.3 Synchronicity with Singularity

As similar to discontinuous dynamical systems in Luo [2–4], the above synchronicity of two dynamical systems in Eqs. (3.1) and (3.2) with constraint in Eq. (3.3) can be extended to the case of higher order singularity. The corresponding definitions can be presented. The definition for the $(2k_\alpha : 2k_\beta)$ synchronization of two dynamical systems in Eqs. (3.1) and (3.2) with constraint in Eq. (3.3) at the corresponding constraint boundary for time $t_m \in [t_{m_1}, t_{m_2}]$ is presented first.

Definition 3.21 Consider two dynamical systems in Eqs. (3.1) and (3.2) with constraint in Eq. (3.3). For $\mathbf{y}_{m\pm}^{(\alpha)} \in \Omega_\alpha$ $(\alpha \in \{1,2\})$ and $\mathbf{y}_m^{(0)} \in \partial\Omega_{12}$ at time t_m, $\mathbf{y}_{m\pm}^{(\alpha)} = \mathbf{y}_m^{(0)}$. For any small $\varepsilon > 0$, there is a time interval $[t_{m-\varepsilon}, t_{m+\varepsilon}]$. At $\mathbf{y}^{(\alpha)} \in \Omega_\alpha^{-\varepsilon}$ for time $t \in [t_{m-\varepsilon}, t_m)$, the constraint function $\varphi(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\lambda})$ is C^{r_α}-continuous $(r_\alpha \geq 2k_\alpha + 1)$ and $|\varphi^{(r_\alpha+1)}(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\lambda})| < \infty$. The two dynamical systems in Eqs. (3.1) and (3.2) with constraint in Eq. (3.3) is called to be synchronized with the $(2k_1 : 2k_2)$-type to the constraint in Eq. (3.3) for time $t_m \in [t_{m_1}, t_{m_2}]$ if for $\alpha = 1, 2$

$$\begin{aligned}
&\varphi(\mathbf{y}_{m-}^{(\alpha)}, t_{m-}, \boldsymbol{\lambda}) = \varphi(\mathbf{y}_{m}^{(0)}, t_{m}, \boldsymbol{\lambda}) = 0;\\
&\varphi^{(s_\alpha)}(\mathbf{y}_{m-}^{(\alpha)}, t_{m-}, \boldsymbol{\lambda}) = 0 \text{ for } s_\alpha = 1, 2, \cdots, 2k_\alpha;\\
&(-1)^{\alpha}[\varphi(\mathbf{y}_{m-\varepsilon}^{(\alpha)}, t_{m-\varepsilon}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}_{m-}^{(\alpha)}, t_{m-}, \boldsymbol{\lambda})] < 0.
\end{aligned} \tag{3.41}$$

As in the definition for the $(2k_1 : 2k_2)$-synchronization, the definition for the $(2k_1 : 2k_2)$-desynchronization of two dynamical systems in Eqs. (3.1) and (3.2) with constraint in Eq. (3.3) on the corresponding constraint boundary for time $t_m \in [t_{m_1}, t_{m_2}]$ is also presented.

Definition 3.22 Consider two dynamical systems in Eqs. (3.1) and (3.2) with constraint in Eq. (3.3). For $\mathbf{y}_{m\pm}^{(\alpha)} \in \Omega_\alpha$ $(\alpha \in \{1,2\})$ and $\mathbf{y}_m^{(0)} \in \partial\Omega_{12}$ at time t_m, $\mathbf{y}_{m\pm}^{(\alpha)} = \mathbf{y}_m^{(0)}$. For any small $\varepsilon > 0$, there is a time interval $[t_{m-\varepsilon}, t_{m+\varepsilon}]$. At $\mathbf{y}^{(\alpha)} \in \Omega_\alpha^{+\varepsilon}$ for time

$t \in (t_{m+}, t_{m+\varepsilon}]$, the constraint function $\varphi(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\lambda})$ is C^{r_α}-continuous $(r_\alpha \geq 2k_\alpha + 1)$ and $|\varphi^{(r_\alpha+1)}(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\lambda})| < \infty$. The two dynamical systems in Eqs. (3.1) and (3.2) with constraint in Eq. (3.3) is said to be desynchronized (or repelled) with the $(2k_1 : 2k_2)$-type to the constraint in Eq. (3.3) for $t_m \in [t_{m_1}, t_{m_2}]$ if for $\alpha = 1, 2$

$$\begin{aligned}
&\varphi(\mathbf{y}_{m+}^{(\alpha)}, t_{m+}, \boldsymbol{\lambda}) = \varphi(\mathbf{y}_m^{(0)}, t_m, \boldsymbol{\lambda}) = 0,\\
&\varphi^{(s_\alpha)}(\mathbf{y}_{m+}^{(\alpha)}, t_{m+}, \boldsymbol{\lambda}) = 0, s_\alpha = 1, 2, \cdots, 2k_\alpha,\\
&(-1)^\alpha[\varphi(\mathbf{y}_{m+\varepsilon}^{(\alpha)}, t_{m+\varepsilon}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}_{m+}^{(\alpha)}, t_{m+}, \boldsymbol{\lambda})] < 0.
\end{aligned} \tag{3.42}$$

As discussed before, the penetration on the boundary of constraint is composed of the semi-synchronization and semi-desynchronization. From the foregoing two definitions, the $(2k_\alpha : 2k_\beta)$-penetration of two dynamical systems in Eqs. (3.1) and (3.2) to constraint in Eq. (3.3) at time t_m is described.

Definition 3.23 Consider two dynamical systems in Eqs. (3.1) and (3.2) with constraint in Eq. (3.3). For $\mathbf{y}_{m\pm}^{(\alpha)} \in \Omega_\alpha$ $(\alpha \in \{1, 2\})$ and $\mathbf{y}_m^{(0)} \in \partial\Omega_{12}$ at time t_m, $\mathbf{y}_{m\pm}^{(\alpha)} = \mathbf{y}_m^{(0)}$. For any small $\varepsilon > 0$, there is a time interval $[t_{m-\varepsilon}, t_{m+\varepsilon}]$. At $\mathbf{y}^{(\alpha)} \in \Omega_\alpha^{\pm\varepsilon}$ for time $t \in [t_{m-\varepsilon}, t_{m+\varepsilon}]$, the constraint function $\varphi(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\lambda})$ is C^{r_α}-continuous $(r_\alpha \geq 2k_\alpha)$ and $|\varphi^{(r_\alpha+1)}(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\lambda})| < \infty$. A flow of two dynamical systems in Eqs. (3.1) and (3.2) with constraint in Eq. (3.3) is said to be *penetrated with the* $(2k_\alpha : 2k_\beta)$*-type* from domain Ω_α to domain Ω_β at the constraint boundary at time t_m if

$$\begin{aligned}
&\varphi(\mathbf{y}_{m-}^{(\alpha)}, t_{m-}, \boldsymbol{\lambda}) = \varphi(\mathbf{y}_{m+}^{(\beta)}, t_{m+}, \boldsymbol{\lambda}) = \varphi(\mathbf{y}_m^{(0)}, t_m, \boldsymbol{\lambda}) = 0;\\
&\varphi^{(s_\alpha)}(\mathbf{y}_{m-}^{(\alpha)}, t_{m-}, \boldsymbol{\lambda}) = 0 \text{ for } s_\alpha = 1, 2, \cdots, 2k_\alpha;\\
&\varphi^{(s_\beta)}(\mathbf{y}_{m+}^{(\beta)}, t_{m+}, \boldsymbol{\lambda}) = 0 \text{ for } s_\beta = 1, 2, \cdots, 2k_\beta;\\
&(-1)^\alpha[\varphi(\mathbf{y}_{m-\varepsilon}^{(\alpha)}, t_{m-\varepsilon}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}_{m-}^{(\alpha)}, t_{m-}, \boldsymbol{\lambda})] < 0 \text{ for } \alpha \in \{1, 2\} \text{ and}\\
&(-1)^\beta[\varphi(\mathbf{y}_{m+\varepsilon}^{(\beta)}, t_{m+\varepsilon}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}_{m+}^{(\beta)}, t_{m+}, \boldsymbol{\lambda})] < 0 \text{ for } \alpha \neq \beta \in \{1, 2\}.
\end{aligned} \tag{3.43}$$

From the three definitions, the higher singularity is used for description of the synchronization, desynchronization, and penetration at the constraint boundary, and the switching among the three synchronous states can be discussed through the higher order singularity as well.

3.4 Higher Order Singularity

From the previous descriptions of the synchronization, desynchronization, and penetration with the higher order singularity for two dynamical systems to the constraint, the higher order singularity of the two dynamical systems to the constraint boundary is discussed as follows.

Definition 3.24 Consider two dynamical systems in Eqs. (3.1) and (3.2) with constraint in Eq. (3.3). For $\mathbf{y}_{m\pm}^{(\alpha)} \in \Omega_\alpha$ $(\alpha \in \{1,2\})$ and $\mathbf{y}_m^{(0)} \in \partial\Omega_{12}$ at time t_m, $\mathbf{y}_{m\pm}^{(\alpha)} = \mathbf{y}_m^{(0)}$. For any small $\varepsilon > 0$, there is a time interval $[t_{m-\varepsilon}, t_{m+\varepsilon}]$. At $\mathbf{y}^{(\alpha)} \in \Omega_\alpha^{\pm\varepsilon}$ for time $t \in [t_{m-\varepsilon}, t_{m+\varepsilon}]$, the constraint function $\varphi(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\lambda})$ is C^{r_α}-continuous $(r_\alpha \geq 2k_\alpha)$ and $|\varphi^{(r_\alpha+1)}(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\lambda})| < \infty$. A resultant flow of the two dynamical systems in Eqs. (3.1) and (3.2) with constraint in Eq. (3.3) is said to be tangential to the constraint boundary with the $(2k_\alpha - 1)$th-order at time t_m if for $\alpha \in \{1,2\}$

$$
\begin{aligned}
&\varphi(\mathbf{y}_{m\pm}^{(\alpha)}, t_{m\pm}, \boldsymbol{\lambda}) = \varphi(\mathbf{y}_m^{(0)}, t_m, \boldsymbol{\lambda}) = 0;\\
&\varphi^{(s_\alpha)}(\mathbf{y}_{m\pm}^{(\alpha)}, t_{m\pm}, \boldsymbol{\lambda}) = 0 \ s_\alpha = 1, 2, \cdots, 2k_\alpha - 1;\\
&(-1)^\alpha[\varphi(\mathbf{y}_{m-\varepsilon}^{(\alpha)}, t_{m-\varepsilon}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}_{m-}^{(\alpha)}, t_{m-}, \boldsymbol{\lambda})] < 0,\\
&(-1)^\alpha[\varphi(\mathbf{y}_{m+\varepsilon}^{(\alpha)}, t_{m+\varepsilon}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}_{m+}^{(\alpha)}, t_{m+}, \boldsymbol{\lambda})] < 0.
\end{aligned} \tag{3.44}
$$

The foregoing definition gives the definition of the $(2k_\alpha - 1)$th tangential condition to the constraint boundary. Based on the similar ideas, the switchability of the synchronization, desynchronization, and penetration of two dynamical systems to the constraint boundary can be described.

Definition 3.25 Consider two dynamical systems in Eqs. (3.1) and (3.2) with constraint in Eq. (3.3). For $\mathbf{y}_{m\pm}^{(\alpha)} \in \Omega_\alpha$ $(\alpha \in \{1,2\})$ and $\mathbf{y}_m^{(0)} \in \partial\Omega_{12}$ at time t_m, $\mathbf{y}_{m\pm}^{(\alpha)} = \mathbf{y}_m^{(0)}$. For any small $\varepsilon > 0$, there is a time interval $[t_{m-\varepsilon}, t_{m+\varepsilon}]$. At $\mathbf{y}^{(\alpha)} \in \Omega_\alpha^{\pm\varepsilon}$ for time $t \in [t_{m-\varepsilon}, t_{m+\varepsilon}]$, the constraint function $\varphi(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\lambda})$ is C^{r_α}-continuous $(r_\alpha \geq 2k_\alpha + 1)$ and $|\varphi^{(r_\alpha+1)}(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\lambda})| < \infty$.

(i) The $(2k_\alpha : 2k_\beta)$-synchronization of the two dynamical systems in Eqs. (3.1) and (3.2) with constraint in Eq. (3.3) is said to be *vanishing* to form a $(2k_\alpha : 2k_\beta)$-penetration from domain Ω_α to domain Ω_β at the constraint boundary at time t_m if for $\alpha, \beta \in \{1,2\}$ and $\alpha \neq \beta$

$$
\begin{aligned}
&\varphi(\mathbf{y}_{m-}^{(\alpha)}, t_{m-}, \boldsymbol{\lambda}) = \varphi(\mathbf{y}_{m\mp}^{(\beta)}, t_{m\mp}, \boldsymbol{\lambda}) = \varphi(\mathbf{y}_m^{(0)}, t_m, \boldsymbol{\lambda}) = 0;\\
&\varphi^{(s_\alpha)}(\mathbf{y}_{m-}^{(\alpha)}, t_{m-}, \boldsymbol{\lambda}) = 0 \text{ for } s_\alpha = 1, 2, \cdots, 2k_\alpha,\\
&\varphi^{(s_\beta)}(\mathbf{y}_{m\mp}^{(\beta)}, t_{m\mp}, \boldsymbol{\lambda}) = 0 \text{ for } s_\beta = 1, 2, \cdots, 2k_\beta + 1;\\
&(-1)^\alpha[\varphi(\mathbf{y}_{m-\varepsilon}^{(\alpha)}, t_{m-\varepsilon}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}_{m-}^{(\alpha)}, t_{m-}, \boldsymbol{\lambda})] < 0,\\
&(-1)^\beta[\varphi(\mathbf{y}_{m\mp\varepsilon}^{(\beta)}, t_{m\mp\varepsilon}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}_{m\mp}^{(\beta)}, t_{m\mp}, \boldsymbol{\lambda})] < 0.
\end{aligned} \tag{3.45}
$$

(ii) The $(2k_\alpha : 2k_\beta)$-synchronization of the two dynamical systems in Eqs. (3.1) and (3.2) with constraint in Eq. (3.3) is said to be *onset* from the $(2k_\alpha : 2k_\beta)$-penetration from Ω_α to Ω_β at the constraint boundary at time t_m if for $\alpha, \beta \in \{1,2\}$ and $\alpha \neq \beta$

$$\begin{aligned}
&\varphi(\mathbf{y}_{m-}^{(\alpha)}, t_{m-}, \boldsymbol{\lambda}) = \varphi(\mathbf{y}_{m\pm}^{(\beta)}, t_{m\pm}, \boldsymbol{\lambda}) = \varphi(\mathbf{y}_{m}^{(0)}, t_m, \boldsymbol{\lambda}) = 0;\\
&\varphi^{(s_\alpha)}(\mathbf{y}_{m-}^{(\alpha)}, t_{m-}, \boldsymbol{\lambda}) = 0 \quad \text{for } s_\alpha = 1, 2, \cdots, 2k_\alpha,\\
&\varphi^{(s_\beta)}(\mathbf{y}_{m\pm}^{(\beta)}, t_{m\pm}, \boldsymbol{\lambda}) = 0 \quad \text{for } s_\beta = 1, 2, \cdots, 2k_\beta + 1;\\
&(-1)^{\alpha}[\varphi(\mathbf{y}_{m-\varepsilon}^{(\alpha)}, t_{m-\varepsilon}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}_{m-}^{(\alpha)}, t_{m-}, \boldsymbol{\lambda})] < 0,\\
&(-1)^{\beta}[\varphi(\mathbf{y}_{m\pm\varepsilon}^{(\beta)}, t_{m\pm\varepsilon}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}_{m\pm}^{(\beta)}, t_{m\pm}, \boldsymbol{\lambda})] < 0.
\end{aligned} \tag{3.46}$$

From this definition, the condition in Eq. (3.45) for *the onset of the* $(2k_\alpha : 2k_\beta)$ *-synchronization* from the $(2k_\alpha : 2k_\beta)$-penetration on the constraint boundary can also be called *the vanishing condition of the* $(2k_\alpha : 2k_\beta)$*-penetration* to form a new $(2k_\alpha : 2k_\beta)$-synchronization on the constraint boundary. In Eq. (3.46), the *vanishing condition of the* $(2k_\alpha : 2k_\beta)$-synchronization to form a new $(2k_\alpha : 2k_\beta)$-penetration can also be called *the onset condition of the* $(2k_\alpha : 2k_\beta)$-penetration from the synchronization. The onset and vanishing conditions of the $(2k_\alpha : 2k_\beta)$-desynchronization from the $(2k_\alpha : 2k_\beta)$-penetration can be discussed. The following definition will give the onset and vanishing conditions of the $(2k_\alpha : 2k_\beta)$-desynchronization.

Definition 3.26 Consider two dynamical systems in Eqs. (3.1) and (3.2) with constraint in Eq. (3.3). For $\mathbf{y}_{m\pm}^{(\alpha)} \in \Omega_\alpha$ $(\alpha \in \{1, 2\})$ and $\mathbf{y}_m^{(0)} \in \partial\Omega_{12}$ at time t_m, $\mathbf{y}_{m\pm}^{(\alpha)} = \mathbf{y}_m^{(0)}$. For any small $\varepsilon > 0$, there is a time interval $[t_{m-\varepsilon}, t_{m+\varepsilon}]$. At $\mathbf{y}^{(\alpha)} \in \Omega_\alpha^{\pm\varepsilon}$ for time $t \in [t_{m-\varepsilon}, t_{m+\varepsilon}]$, the constraint function $\varphi(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\lambda})$ is C^{r_α}-continuous $(r_\alpha \geq 2k_\alpha + 1)$ and $|\varphi^{(r_\alpha+1)}(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\lambda})| < \infty$.

(i) The $(2k_\alpha : 2k_\beta)$-synchronization of the two dynamical systems in Eqs. (3.1) and (3.2) with constraint in Eq. (3.3) is called to be *vanishing* to form a $(2k_\alpha : 2k_\beta)$-desynchronization at the constraint boundary at time t_m if for $\alpha, \beta \in \{1,2\}$ and $\alpha \neq \beta$

$$\begin{aligned}
&\varphi(\mathbf{y}_{m\mp}^{(\alpha)}, t_{m\mp}, \boldsymbol{\lambda}) = \varphi(\mathbf{y}_{m\mp}^{(\beta)}, t_{m\mp}, \boldsymbol{\lambda}) = \varphi(\mathbf{y}_{m}^{(0)}, t_m, \boldsymbol{\lambda}) = 0,\\
&\varphi^{(s_\alpha)}(\mathbf{y}_{m\mp}^{(\alpha)}, t_{m\mp}, \boldsymbol{\lambda}) = 0 \quad \text{for } s_\alpha = 1, 2, \cdots, 2k_\alpha + 1,\\
&\varphi^{(s_\beta)}(\mathbf{y}_{m\mp}^{(\beta)}, t_{m\mp}, \boldsymbol{\lambda}) = 0 \quad \text{for } s_\beta = 1, 2, \cdots, 2k_\beta + 1,\\
&(-1)^{\alpha}[\varphi(\mathbf{y}_{m\mp\varepsilon}^{(\alpha)}, t_{m\mp\varepsilon}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}_{m\mp}^{(\alpha)}, t_{m\mp}, \boldsymbol{\lambda})] < 0,\\
&(-1)^{\beta}[\varphi(\mathbf{y}_{m\mp\varepsilon}^{(\beta)}, t_{m\mp\varepsilon}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}_{m\mp}^{(\beta)}, t_{m\mp}, \boldsymbol{\lambda})] < 0.
\end{aligned} \tag{3.47}$$

(ii) The $(2k_\alpha : 2k_\beta)$-synchronization of the two dynamical systems in Eqs. (3.1) and (3.2) with constraint in Eq. (3.3) is said to be *onset* from the $(2k_\alpha : 2k_\beta)$-desynchronization at the constraint boundary at time t_m if for $\alpha, \beta \in \{1, 2\}$ and $\alpha \neq \beta$

$$\begin{aligned}
&\varphi(\mathbf{y}_{m\pm}^{(\alpha)}, t_{m\pm}, \boldsymbol{\lambda}) = \varphi(\mathbf{y}_{m\pm}^{(\beta)}, t_{m\pm}, \boldsymbol{\lambda}) = \varphi(\mathbf{y}_{m}^{(0)}, t_m, \boldsymbol{\lambda}) = 0,\\
&\varphi^{(s_\alpha)}(\mathbf{y}_{m\pm}^{(\alpha)}, t_{m\pm}, \boldsymbol{\lambda}) = 0 \quad \text{for } s_\alpha = 1, 2, \cdots, 2k_\alpha + 1,\\
&\varphi^{(s_\beta)}(\mathbf{y}_{m\pm}^{(\beta)}, t_{m\pm}, \boldsymbol{\lambda}) = 0 \quad \text{for } s_\beta = 1, 2, \cdots, 2k_\beta + 1,\\
&(-1)^{\alpha}[\varphi(\mathbf{y}_{m\pm\varepsilon}^{(\alpha)}, t_{m\pm\varepsilon}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}_{m\pm}^{(\alpha)}, t_{m\pm}, \boldsymbol{\lambda})] < 0,\\
&(-1)^{\beta}[\varphi(\mathbf{y}_{m\pm\varepsilon}^{(\beta)}, t_{m\pm\varepsilon}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}_{m\pm}^{(\beta)}, t_{m\pm}, \boldsymbol{\lambda})] < 0.
\end{aligned} \tag{3.48}$$

The conditions in Eqs. (3.47) and (3.48) are inversely switched. The condition in Eq. (3.47) for *the onset condition of the* $(2k_\alpha : 2k_\beta)$*-synchronization* from the $(2k_\alpha : 2k_\beta)$-desynchronization on the constraint boundary can be called *the vanishing condition of the* $(2k_\alpha : 2k_\beta)$-desynchronization to form a new $(2k_\alpha : 2k_\beta)$-synchronization on such a constraint boundary. However, the condition in Eq. (3.48) for *the vanishing condition of the* $(2k_\alpha : 2k_\beta)$-synchronization to form a new $(2k_\alpha : 2k_\beta)$-penetration can be called *the onset condition of the* $(2k_\alpha : 2k_\beta)$-desynchronization from the synchronization. The switching of desynchronization and penetration on the boundary will be discussed as follows.

Definition 3.27 Consider two dynamical systems in Eqs. (3.1) and (3.2) with constraint in Eq. (3.3). For $\mathbf{y}_{m\pm}^{(\alpha)} \in \Omega_\alpha$ $(\alpha \in \{1,2\})$ and $\mathbf{y}_m^{(0)} \in \partial\Omega_{12}$ at time t_m, $\mathbf{y}_{m\pm}^{(\alpha)} = \mathbf{y}_m^{(0)}$. For any small $\varepsilon > 0$, there is a time interval $[t_{m-\varepsilon}, t_{m+\varepsilon}]$. At $\mathbf{y}^{(\alpha)} \in \Omega_\alpha^{\pm\varepsilon}$ for time $t \in [t_{m-\varepsilon}, t_{m+\varepsilon}]$, the constraint function $\varphi(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\lambda})$ is C^{r_α}-continuous $(r_\alpha \geq 2k_\alpha + 1)$ and $|\varphi^{(r_\alpha+1)}(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\lambda})| < \infty$.

(i) The $(2k_\alpha : 2k_\beta)$-desynchronization of the two dynamical systems in Eqs. (3.1) and (3.2) with constraint in Eq. (3.3) is called to be *vanishing* to form a $(2k_\alpha : 2k_\beta)$-penetration from domain Ω_α to domain Ω_β at the constraint boundary at time t_m if for $\alpha, \beta \in \{1,2\}$ and $\alpha \neq \beta$

$$\begin{aligned}
&\varphi(\mathbf{y}_{m\pm}^{(\alpha)}, t_{m\pm}, \boldsymbol{\lambda}) = \varphi(\mathbf{y}_{m+}^{(\beta)}, t_{m+}, \boldsymbol{\lambda}) = \varphi(\mathbf{y}_m^{(0)}, t_m, \boldsymbol{\lambda}) = 0,\\
&\varphi^{(s_\alpha)}(\mathbf{y}_{m\pm}^{(\alpha)}, t_{m\pm}, \boldsymbol{\lambda}) = 0 \quad \text{for } s_\alpha = 1, 2, \cdots, 2k_\alpha + 1,\\
&\varphi^{(s_\beta)}(\mathbf{y}_{m+}^{(\beta)}, t_{m+}, \boldsymbol{\lambda}) = 0 \quad \text{for } s_\beta = 1, 2, \cdots, 2k_\beta,\\
&(-1)^\alpha[\varphi(\mathbf{y}_{m\pm\varepsilon}^{(\alpha)}, t_{m\pm\varepsilon}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}_{m\pm}^{(\alpha)}, t_{m\pm}, \boldsymbol{\lambda})] < 0,\\
&(-1)^\beta[\varphi(\mathbf{y}_{m+\varepsilon}^{(\beta)}, t_{m+\varepsilon}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}_{m+}^{(\beta)}, t_{m+}, \boldsymbol{\lambda})] < 0.
\end{aligned} \tag{3.49}$$

(ii) The $(2k_\alpha : 2k_\beta)$-desynchronization of the two dynamical systems in Eqs. (3.1) and (3.2) with constraint in Eq. (3.3) is said to be *onset* from the $(2k_\alpha : 2k_\beta)$-penetration from domain Ω_α to domain Ω_β at the constraint boundary at time t_m if for $\alpha, \beta \in \{1,2\}$ and $\alpha \neq \beta$

$$\begin{aligned}
&\varphi(\mathbf{y}_{m\mp}^{(\alpha)}, t_{m\mp}, \boldsymbol{\lambda}) = \varphi(\mathbf{y}_{m+}^{(\beta)}, t_{m+}, \boldsymbol{\lambda}) = \varphi(\mathbf{y}_m^{(0)}, t_m, \boldsymbol{\lambda}) = 0,\\
&\varphi^{(s_\alpha)}(\mathbf{y}_{m\mp}^{(\alpha)}, t_{m\mp}, \boldsymbol{\lambda}) = 0 \quad \text{for } s_\alpha = 1, 2, \cdots, 2k_\alpha + 1,\\
&\varphi^{(s_\beta)}(\mathbf{y}_{m+}^{(\beta)}, t_{m+}, \boldsymbol{\lambda}) = 0 \quad \text{for } s_\beta = 1, 2, \cdots, 2k_\beta,\\
&(-1)^\alpha[\varphi(\mathbf{y}_{m\mp\varepsilon}^{(\alpha)}, t_{m\mp\varepsilon}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}_{m\mp}^{(\alpha)}, t_{m\mp}, \boldsymbol{\lambda})] < 0,\\
&(-1)^\beta[\varphi(\mathbf{y}_{m+\varepsilon}^{(\beta)}, t_{m+\varepsilon}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}_{m+}^{(\beta)}, t_{m+}, \boldsymbol{\lambda})] < 0.
\end{aligned} \tag{3.50}$$

In Eq. (3.49), *the onset condition of the* $(2k_\alpha : 2k_\beta)$*-desynchronization* from the $(2k_\alpha : 2k_\beta)$-penetration on the constraint boundary can be called *the vanishing condition of the* $(2k_\alpha : 2k_\beta)$-penetration to form a new $(2k_\alpha : 2k_\beta)$−desynchronization on the constraint boundary. However, in Eq. (3.50), *the vanishing condition of the* $(2k_\alpha : 2k_\beta)$-synchronization to form a new $(2k_\alpha : 2k_\beta)$-penetration can be called *the onset condition of the* $(2k_\alpha : 2k_\beta)$-penetration from the $(2k_\alpha : 2k_\beta)$-desynchronization.

Definition 3.28 Consider two dynamical systems in Eqs. (3.1) and (3.2) with constraint in Eq. (3.3). For $\mathbf{y}_{m\pm}^{(\alpha)} \in \Omega_\alpha\,(\alpha \in \{1,2\})$ and $\mathbf{y}_m^{(0)} \in \partial\Omega_{12}$ at time t_m, $\mathbf{y}_{m\pm}^{(\alpha)} = \mathbf{y}_m^{(0)}$. For any small $\varepsilon > 0$, there is a time interval $[t_{m-\varepsilon}, t_{m+\varepsilon}]$. At $\mathbf{y}^{(\alpha)} \in \Omega_\alpha^{\pm\varepsilon}$ for time $t \in [t_{m-\varepsilon}, t_{m+\varepsilon}]$, the constraint function $\varphi(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\lambda})$ is C^{r_α}-continuous $(r_\alpha \geq 2k_\alpha + 1)$ and $|\varphi^{(r_\alpha+1)}(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\lambda})| < \infty$. The $(2k_\alpha : 2k_\beta)$-penetration of the two dynamical systems in Eqs. (3.1) and (3.2) with constraint in Eq. (3.3) is called to be *switched* to a new $(2k_\beta : 2k_\alpha)$-penetration at the constraint boundary at time t_m if for $\alpha, \beta \in \{1,2\}$ and $\alpha \neq \beta$

$$
\begin{aligned}
&\varphi(\mathbf{y}_{m\mp}^{(\alpha)}, t_{m\mp}, \boldsymbol{\lambda}) = \varphi(\mathbf{y}_{m\pm}^{(\beta)}, t_{m\pm}, \boldsymbol{\lambda}) = \varphi(\mathbf{y}_m^{(0)}, t_m, \boldsymbol{\lambda}) = 0,\\
&\varphi^{(s_\alpha)}(\mathbf{y}_{m\mp}^{(\alpha)}, t_{m\mp}, \boldsymbol{\lambda}) = 0 \text{ for } s_\alpha = 1, 2, \cdots, 2k_\alpha + 1,\\
&\varphi^{(s_\beta)}((\mathbf{y}_{m\pm}^{(\beta)}, t_{m\pm}, \boldsymbol{\lambda}) = 0 \text{ for } s_\beta = 1, 2, \cdots, 2k_\beta + 1,\\
&(-1)^\alpha[\varphi(\mathbf{y}_{m\mp\varepsilon}^{(\alpha)}, t_{m\mp\varepsilon}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}_{m\mp}^{(\alpha)}, t_{m\mp}, \boldsymbol{\lambda})] < 0,\\
&(-1)^\beta[\varphi(\mathbf{y}_{m\pm\varepsilon}^{(\beta)}, t_{m\pm\varepsilon}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}_{m\pm}^{(\beta)}, t_{m\pm}, \boldsymbol{\lambda})] < 0.
\end{aligned} \tag{3.51}
$$

In the foregoing definition, the condition for the $(2k_\alpha : 2k_\beta)$-penetration switching to the $(2k_\beta : 2k_\alpha)$-penetration at the boundary is presented.

3.5 Synchronization to Constraint

In the previous section, the definitions for the synchronicity and the corresponding singularity of two dynamical systems to the constraint were discussed. What conditions can guarantee such synchronicity of the two dynamical systems to the constraint exists? In this section, necessary and sufficient conditions for the synchronization of two dynamical systems to the specific constraint will be presented. The synchronicity switching is discussed through the singularity of a flow of the resultant system to the constraint boundary.

Theorem 3.1 *Consider two dynamical systems in Eqs. (3.1) and (3.2) with constraint in Eq. (3.3). For* $\mathbf{y}_{m\pm}^{(\alpha)} \in \Omega_\alpha\,(\alpha \in \{1,2\})$ *and* $\mathbf{y}_m^{(0)} \in \partial\Omega_{12}$ *at time* t_m, $\mathbf{y}_{m\pm}^{(\alpha)} = \mathbf{y}_m^{(0)}$. *For any small* $\varepsilon > 0$, *there is a time interval* $[t_{m-\varepsilon}, t_{m+\varepsilon}]$. *At* $\mathbf{y}^{(\alpha)} \in \Omega_\alpha^{\pm\varepsilon}$ *for time* $t \in [t_{m-\varepsilon}, t_{m+\varepsilon}]$, *the constraint function* $\varphi(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\lambda})$ *is* C^{r_α}-*continuous* $(r_\alpha \geq 3)$ *and* $|\varphi^{(r_\alpha+1)}(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\lambda})| < \infty$. *For* $\mathbf{y}^{(\alpha)} \in \Omega_\alpha$ *and* $\mathbf{y}^{(0)} \in \partial\Omega_{12}$, *suppose* $D^{s_\alpha}\mathbb{F}^{(\alpha)}(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\pi}^{(\alpha)}) = D^{s_\alpha}\mathbb{F}^{(0)}(\mathbf{y}^{(0)}, t, \boldsymbol{\lambda})$ *(*$s_\alpha = 0, 1, 2, \cdots$*) for* $\mathbf{y}^{(\alpha)} = \mathbf{y}^{(0)}$. *The two dynamical systems in Eqs. (3.1) and (3.2) to the constraint in Eq. (3.3) are synchronized for time* $t \in [t_{m_1}, t_{m_2}]$ *if and only if*

(i) *for* $\mathbf{y}_{m\pm}^{(\alpha)} \in \Omega_\alpha$ *and* $\mathbf{y}_m^{(0)} \in \partial\Omega_{12}$ *with any time* t_m

$$\begin{aligned}&\mathbf{y}_{m\pm}^{(\alpha)} = \mathbf{y}_m^{(0)}, \varphi^{(r_\alpha)}(\mathbf{y}_{m\pm}^{(\alpha)}, t_m, \boldsymbol{\lambda}) = 0\\ &\text{for } \alpha = 1,2 \text{ and } r_\alpha = 0,1,2,\cdots;\end{aligned} \tag{3.52}$$

(ii) *for* $\mathbf{y}_\kappa^{(\alpha)} \in \Omega_\alpha^{-\varepsilon}$ *at time* $t_\kappa^- \in [t_{m-\varepsilon}, t_m)$ *and* $\mathbf{y}_m^{(0)} \in \partial\Omega_{12}$ *with* $t_m \in (t_{m_1}, t_{m_2})$

$$\begin{aligned}&\mathbf{y}_\kappa^{(\alpha)} \neq \mathbf{y}_m^{(0)}, (-1)^\alpha \varphi^{(1)}(\mathbf{y}_\kappa^{(\alpha)}, t_\kappa^-, \boldsymbol{\lambda}) > 0,\\ &\lim_{t_\kappa^- \to t_m} \varphi^{(1)}(\mathbf{y}_\kappa^{(\alpha)}, t_\kappa^-, \boldsymbol{\lambda}) = 0 \text{ for } \alpha = 1,2;\end{aligned} \tag{3.53}$$

(iii) *for* $\mathbf{y}_\kappa^{(\alpha)} \in \Omega_\alpha^{+\varepsilon}$ *at time* $t_\kappa^+ \in (t_m, t_{m+\varepsilon}]$ *and* $\mathbf{y}_m^{(0)} \in \partial\Omega_{12}$ *with* $t_m \notin [t_{m_1}, t_{m_2}]$

$$\begin{aligned}&\mathbf{y}_\kappa^{(\alpha)} \neq \mathbf{y}_m^{(0)}, (-1)^\alpha \varphi^{(1)}(\mathbf{y}_\kappa^{(\alpha)}, t_\kappa^+, \boldsymbol{\lambda}) < 0,\\ &\lim_{t_\kappa^+ \to t_m} \varphi^{(1)}(\mathbf{y}_\kappa^{(\alpha)}, t_\kappa^+, \boldsymbol{\lambda}) = 0 \text{ for } \alpha = 1,2;\end{aligned} \tag{3.54}$$

(iv) *for* $\mathbf{y}_\kappa^{(\alpha)} \in \Omega_\alpha^{\pm\varepsilon}$ *at time* $t_\kappa^- \in [t_{m-\varepsilon}, t_{m-})$ *or* $t_\kappa^+ \in (t_{m+}, t_{m+\varepsilon}]$ *and* $\mathbf{y}_m^{(0)} \in \partial\Omega_{12}$ *with* $t_m = t_{m_1}$ and t_{m_2}

$$\begin{aligned}&\mathbf{y}_\kappa^{(\alpha)} \neq \mathbf{y}_m^{(0)}, \lim_{t_\kappa^\pm \to t_{m\pm}} \varphi^{(1)}(\mathbf{y}_\kappa^{(\alpha)}, t_\kappa^\pm, \boldsymbol{\lambda}) = 0,\\ &\lim_{t_\kappa^\pm \to t_{m\pm}} (-1)^\alpha \varphi^{(2)}(\mathbf{y}_\kappa^{(\alpha)}, t_\kappa^\pm, \boldsymbol{\lambda}) < 0 \text{ for } \alpha = 1,2;\end{aligned} \tag{3.55}$$

Proof (i) Consider two dynamical systems in Eqs. (3.1) and (3.2) with a constraint condition in Eq. (3.3). From Definition 3.10, one has for $\mathbf{y}^{(\alpha)} = \mathbf{y}^{(0)} \in \partial\Omega_{12}$,

$$\varphi(\mathbf{y}^{(\alpha)}(t), t, \boldsymbol{\lambda}) = \varphi(\mathbf{y}^{(0)}(t), t, \boldsymbol{\lambda}) = 0.$$

Because $D^{s_\alpha}\mathbb{F}^{(\alpha)}(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\pi}^{(\alpha)}) = D^{s_\alpha}\mathbb{F}^{(0)}(\mathbf{y}^{(0)}, t, \boldsymbol{\lambda})$ ($s_\alpha = 0,1,2,\cdots$) on the constraint boundary $\partial\Omega_{12}$, one obtains $d^{r_\alpha}\mathbf{y}^{(\alpha)}/dt^{r_\alpha} = d^{r_\alpha}\mathbf{y}^{(0)}(t)/dt^{r_\alpha}$ ($r_\alpha = 1,2,3,\cdots$). The foregoing equation gives

$$\varphi^{(r_\alpha)}(\mathbf{y}^{(\alpha)}(t), t, \boldsymbol{\lambda}) = \varphi^{(r_\alpha)}(\mathbf{y}^{(0)}(t), t, \boldsymbol{\lambda}) = 0.$$

(ii) and (iii) For $\mathbf{y}_\kappa^{(\alpha)} \in \Omega_\alpha^{\pm\varepsilon}$ at time $t_\kappa^- \in [t_{m-\varepsilon}, t_m)$ or $t_\kappa^+ \in (t_m, t_{m+\varepsilon}]$ and $\mathbf{y}_m^{(0)} \in \partial\Omega_{12}$ with $t_m \in (t_{m_1}, t_{m_2})$,

$$\varphi(\mathbf{y}_\kappa^{(1)}, t_\kappa^\pm, \boldsymbol{\lambda}) > 0 \text{ and } \varphi(\mathbf{y}_\kappa^{(2)}, t_\kappa^\pm, \boldsymbol{\lambda}) < 0.$$

Introduce $0 < \varepsilon_1 = |t_{m\pm\varepsilon} - t_\kappa^\pm| < |t_{m\pm\varepsilon} - t_m| = \varepsilon$ for $t_m > t_\kappa^-$ and $t_m < t_\kappa^+$. Because of

$$\varphi(\mathbf{y}_{m\pm\varepsilon}^{(\alpha)}, t_{m\pm\varepsilon}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}_\kappa^{(\alpha)}, t_\kappa^\pm, \boldsymbol{\lambda}) = \varphi^{(1)}(\mathbf{y}_\kappa^{(\alpha)}, t_\kappa^\pm, \boldsymbol{\lambda})(\pm\varepsilon_1) + o(\varepsilon_1)$$

and once higher order terms drop, the foregoing equation leads to

$$\varphi(\mathbf{y}_{m\pm\varepsilon}^{(\alpha)}, t_{m\pm\varepsilon}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}_\kappa^{(\alpha)}, t_\kappa^\pm, \boldsymbol{\lambda}) = \varphi^{(1)}(\mathbf{y}_\kappa^{(\alpha)}, t_\kappa^\pm, \boldsymbol{\lambda})(\pm\varepsilon_1)$$

From Definition 2.13 for $t_m \in (t_{m_1}, t_{m_2})$ with t_κ^-, we have

$$\lim_{t_\kappa^- \to t_{m-}} (-1)^\alpha \varphi^{(1)}(\mathbf{y}_\kappa^{(\alpha)}, t_\kappa^\pm, \boldsymbol{\lambda}) > 0,$$
$$\lim_{t_\kappa^- \to t_m} \varphi^{(1)}(\mathbf{y}_\kappa^{(\alpha)}, t_\kappa^-, \boldsymbol{\lambda}) = \varphi^{(1)}(\mathbf{y}_m^{(\alpha)}, t_m, \boldsymbol{\lambda}) = 0.$$

However, using Eq. (3.53), the condition in Definition 3.13 is obtained.
From Definition 3.14 for $t_m \notin [t_{m_1}, t_{m_2}]$ with t_κ^+, we have

$$\lim_{t_\kappa^+ \to t_{m+}} (-1)^\alpha \varphi^{(1)}(\mathbf{y}_\kappa^{(\alpha)}, t_\kappa^\pm, \boldsymbol{\lambda}) < 0,$$
$$\lim_{t_\kappa^+ \to t_m} \varphi^{(1)}(\mathbf{y}_\kappa^{(\alpha)}, t_\kappa^+, \boldsymbol{\lambda}) = \varphi^{(1)}(\mathbf{y}_m^{(\alpha)}, t_m, \boldsymbol{\lambda}) = 0.$$

However, using Eq. (3.54), the condition in Definition 3.14 is obtained.
(iv) For $\mathbf{y}_\kappa^{(\alpha)} \in \boldsymbol{\Omega}_\alpha^{\pm\varepsilon}$ at time $t_\kappa^- \in [t_{m-\varepsilon}, t_{m-})$ or $t_\kappa^+ \in (t_{m+}, t_{m+\varepsilon}]$ and $\mathbf{y}_m^{(0)} \in \partial\boldsymbol{\Omega}_{12}$ with $t_m = t_{m_1}$ and t_{m_2},

$$\begin{aligned}
&\lim_{t_\kappa^\pm \to t_{m\pm}} [\varphi(\mathbf{y}_{m\pm\varepsilon_1}^{(\alpha)}, t_{m\pm\varepsilon_1}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}_\kappa^{(\alpha)}, t_\kappa^\pm, \boldsymbol{\lambda})] \\
&\quad = \lim_{t_\kappa^\pm \to t_{m\pm}} \varphi^{(1)}(\mathbf{y}_\kappa^{(\alpha)}, t_\kappa^\pm, \boldsymbol{\lambda})(\pm\varepsilon_1) \\
&\quad + \lim_{t_\kappa^\pm \to t_{m\pm}} \frac{1}{2!}\varphi^{(2)}(\mathbf{y}_\kappa^{(\alpha)}, t_\kappa^\pm, \boldsymbol{\lambda})(\pm\varepsilon_1)^2 + o({\varepsilon_1}^2)
\end{aligned}$$

Ignoring the third-order term and the higher order terms of ε_1, we have

$$\begin{aligned}
&\lim_{t_\kappa^\pm \to t_{m\pm}} [\varphi(\mathbf{y}_{m\pm\varepsilon_1}^{(\alpha)}, t_{m\pm\varepsilon_1}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}_\kappa^{(\alpha)}, t_\kappa^\pm, \boldsymbol{\lambda})] \\
&\quad = \lim_{t_\kappa^\pm \to t_{m\pm}} \varphi^{(1)}(\mathbf{y}_\kappa^{(\alpha)}, t_\kappa^\pm, \boldsymbol{\lambda})(\pm\varepsilon_1) \\
&\quad + \lim_{t_\kappa^\pm \to t_{m\pm}} \frac{1}{2!}\varphi^{(2)}(\mathbf{y}_\kappa^{(\alpha)}, t_\kappa^\pm, \boldsymbol{\lambda})(\pm\varepsilon_1)^2
\end{aligned}$$

Using $\lim_{\kappa_i \to m_i\pm} \varphi^{(1)}(\mathbf{y}_\kappa^{(\alpha)}, t_\kappa^\pm, \boldsymbol{\lambda}) = 0$, the foregoing equation gives

$$\begin{aligned}
&\lim_{t_\kappa^\pm \to t_{m\pm}} [\varphi(\mathbf{y}_{m\pm\varepsilon_1}^{(\alpha)}, t_{m\pm\varepsilon_1}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}_\kappa^{(\alpha)}, t_\kappa^\pm, \boldsymbol{\lambda})] \\
&\quad = \lim_{t_\kappa^\pm \to t_{m\pm}} \frac{1}{2!}\varphi^{(2)}(\mathbf{y}_\kappa^{(\alpha)}, t_\kappa^\pm, \boldsymbol{\lambda})(\pm\varepsilon_1)^2
\end{aligned}$$

If $\lim_{t_\kappa^\pm \to t_{m\pm}} (-1)^\alpha \varphi^{(2)}(\mathbf{y}_\kappa^{(\alpha)}, t_\kappa^\pm, \boldsymbol{\lambda}) < 0$, we have

$$\lim_{t_\kappa^\pm \to t_{m\pm}} [\varphi(\mathbf{y}_{m\pm\varepsilon_1}^{(\alpha)}, t_{m\pm\varepsilon_1}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}_\kappa^{(\alpha)}, t_\kappa^\pm, \boldsymbol{\lambda})] < 0$$

From Definition 3.18, the point $(\mathbf{x}_{m_i\pm}^{(\alpha)}, t_{m_i\pm})$ $(i = 1, 2)$ is tangential point to the constraint. The synchronization at such a point appears or disappears. However, from the conditions in Definition 3.18, Eq. (3.55) can be obtained. This theorem is proved. □

For the point $(\mathbf{y}_{m_1}^{(\alpha)}, t_{m_1})$, the synchronization will be onset. However, for the point $(\mathbf{y}_{m_2}^{(\alpha)}, t_{m_2})$, the synchronization will vanish. For $t_m \in (t_{m_1}, t_{m_2})$, the synchronization at point $(\mathbf{y}_m^{(\alpha)}, t_m)$ on the constraint boundary can be formed. For $t_m \notin [t_{m_1}, t_{m_2}]$, the desynchronization at point $(\mathbf{y}_m^{(\alpha)}, t_m)$ on the constraint boundary can be formed. If $t_{m_1} \to -\infty$ and $t_{m_2} \to \infty$, the synchronization is absolute. The synchronization of two dynamical systems to the constraint can occur at any time t_m. Once the synchronization is formed on the constraint boundary, such synchronization on the constraint boundary will not disappear. If the higher order singularity on the boundary exists, the corresponding theorem is presented in a similar fashion.

Theorem 3.2 *Consider two dynamical systems in Eqs. (3.1) and (3.2) with constraint in Eq. (3.3). For* $\mathbf{y}_{m\pm}^{(\alpha)} \in \boldsymbol{\Omega}_\alpha$ $(\alpha \in \{1, 2\})$ *and* $\mathbf{y}_m^{(0)} \in \partial\boldsymbol{\Omega}_{12}$ *at time* $t_m, \mathbf{y}_{m\pm}^{(\alpha)} = \mathbf{y}_m^{(0)}$. *For any small* $\varepsilon > 0$, *there is a time interval* $[t_{m-\varepsilon}, t_{m+\varepsilon}]$. *At* $\mathbf{y}^{(\alpha)} \in \boldsymbol{\Omega}_\alpha^{\pm\varepsilon}$ *for time* $t \in [t_{m-\varepsilon}, t_{m+\varepsilon}]$, *the constraint function* $\varphi(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\lambda})$ *is* C^{r_α}*-continuous* $(r_\alpha \ge 2k_\alpha + 1)$ *and* $|\varphi^{(r_\alpha+1)}(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\lambda})| < \infty$. *For* $\mathbf{y}^{(\alpha)} \in \boldsymbol{\Omega}_\alpha$ *and* $\mathbf{y}^{(0)} \in \partial\boldsymbol{\Omega}_{12}$, *suppose* $D^{s_\alpha}\mathbb{F}^{(\alpha)}(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\pi}^{(\alpha)}) = D^{s_\alpha}\mathbb{F}^{(0)}(\mathbf{y}^{(0)}, t, \boldsymbol{\lambda})$ $(s_\alpha = 0, 1, 2, \cdots)$ *for* $\mathbf{y}^{(\alpha)} = \mathbf{y}^{(0)}$. *The two dynamical systems in Eqs. (3.1) and (3.2) to the constraint in Eq. (3.3) are synchronized for time* $t \in [t_{m_1}, t_{m_2}]$ *if and only if*

(i) *for* $\mathbf{y}_m^{(\alpha)} \in \boldsymbol{\Omega}_\alpha$ *and* $\mathbf{y}_m^{(0)} \in \partial\boldsymbol{\Omega}_{12}$ *with any time* t_m

$$\begin{aligned} &\mathbf{y}_m^{(\alpha)} = \mathbf{y}_m^{(0)}, \varphi^{(r_\alpha)}(\mathbf{y}_m^{(\alpha)}, t_m, \boldsymbol{\lambda}) = 0 \\ &\text{for } \alpha = 1, 2 \text{ and } r_\alpha = 0, 1, 2, \cdots; \end{aligned} \tag{3.56}$$

(ii) *for* $\mathbf{y}_\kappa^{(\alpha)} \in \boldsymbol{\Omega}_\alpha^{-\varepsilon}$ *at time* $t_\kappa^- \in [t_{m-\varepsilon}, t_m)$ *and* $\mathbf{y}_m^{(0)} \in \partial\boldsymbol{\Omega}_{12}$ *with* $t_m \in (t_{m_1}, t_{m_2})$

$$\left.\begin{aligned} &\mathbf{y}_\kappa^{(\alpha)} \neq \mathbf{y}_m^{(0)}, \lim_{t_\kappa^- \to t_{m-}} \varphi^{(s_\alpha)}(\mathbf{y}_\kappa^{(\alpha)}, t_\kappa^-, \boldsymbol{\lambda}) = 0 \text{ for } s_\alpha = 1, 2, \cdots, 2k_\alpha, \\ &\lim_{t_\kappa^- \to t_{m-}} (-1)^\alpha \varphi^{(2k_\alpha+1)}(\mathbf{y}_\kappa^{(\alpha)}, t_\kappa^-, \boldsymbol{\lambda}) > 0, \\ &\lim_{t_\kappa^- \to t_m} \varphi^{(2k_\alpha+1)}(\mathbf{y}_\kappa^{(\alpha)}, t_\kappa^-, \boldsymbol{\lambda}) = 0 \text{ for } \alpha = 1, 2; \end{aligned}\right\} \tag{3.57}$$

(iii) *for* $\mathbf{y}_\kappa^{(\alpha)} \in \Omega_\alpha^{+\varepsilon}$ *at time* $t_\kappa^+ \in (t_m, t_{m+\varepsilon}]$ *and* $\mathbf{y}_m^{(0)} \in \partial\Omega_{12}$ *with* $t_m \notin [t_{m_1}, t_{m_2}]$

$$\left.\begin{aligned} &\mathbf{y}_\kappa^{(\alpha)} \neq \mathbf{y}_m^{(0)},\ \lim_{t_\kappa^+ \to t_{m+}} \varphi^{(s_\alpha)}(\mathbf{y}_\kappa^{(\alpha)}, t_\kappa^+, \boldsymbol{\lambda}) = 0 \text{ for } s_\alpha = 1, 2, \cdots, 2k_\alpha, \\ &\lim_{t_\kappa^+ \to t_{m+}} (-1)^\alpha \varphi^{(2k_\alpha+1)}(\mathbf{y}_\kappa^{(\alpha)}, t_\kappa^+, \boldsymbol{\lambda}) < 0, \\ &\lim_{t_\kappa^+ \to t_m} \varphi^{(2k_\alpha+1)}(\mathbf{y}_\kappa^{(\alpha)}, t_\kappa^+, \boldsymbol{\lambda}) = 0 \text{ for } \alpha = 1, 2; \end{aligned}\right\} \tag{3.58}$$

(iv) *for* $\mathbf{y}_\kappa^{(\alpha)} \in \Omega_\alpha^{+\varepsilon}$ *at time* $t_\kappa^- \in [t_{m-\varepsilon}, t_{m-})$ *or* $t_\kappa^+ \in (t_{m+}, t_{m+\varepsilon}]$ *and* $\mathbf{y}_m^{(0)} \in \partial\Omega_{12}$ *with* $t_m = t_{m_1}$ and t_{m_2}

$$\left.\begin{aligned} &\mathbf{y}_\kappa^{(\alpha)} \neq \mathbf{y}_m^{(0)},\ \lim_{t_\kappa^\pm \to t_{m\pm}} \varphi^{(s_\alpha)}(\mathbf{y}_\kappa^{(\alpha)}, t_\kappa^\pm, \boldsymbol{\lambda}) = 0 \text{ for } s_\alpha = 1, 2, \cdots, 2k_\alpha+1, \\ &\lim_{t_\kappa^\pm \to t_{m\pm}} (-1)^\alpha \varphi^{(2k_\alpha+2)}(\mathbf{y}_\kappa^{(\alpha)}, t_\kappa^\pm, \boldsymbol{\lambda}) < 0 \text{ for } \alpha = 1, 2. \end{aligned}\right\} \tag{3.59}$$

Proof (i) Consider two dynamical systems in Eqs. (3.1) and (3.2) with a constraint condition in Eq. (3.3). From Definition 3.10, one has for $\mathbf{y}^{(\alpha)} = \mathbf{y}^{(0)} \in \partial\Omega_{12}$,

$$\varphi(\mathbf{y}^{(\alpha)}(t), t, \boldsymbol{\lambda}) = \varphi(\mathbf{y}^{(0)}(t), t, \boldsymbol{\lambda}) = 0.$$

Because $D^{s_\alpha}\mathbb{F}^{(\alpha)}(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\pi}^{(\alpha)}) = D^{s_\alpha}\mathbb{F}^{(0)}(\mathbf{y}^{(0)}, t, \boldsymbol{\lambda})(s_\alpha = 0, 1, 2, \cdots)$ on the constraint boundary $\partial\Omega_{12}$, one obtains $d^{r_\alpha}\mathbf{y}^{(\alpha)}/dt^{r_\alpha} = d^{r_\alpha}\mathbf{y}^{(0)}(t)/dt^{r_\alpha}$ $(r_\alpha = 1, 2, 3, \cdots)$. The foregoing equation gives

$$\varphi^{(r_\alpha)}(\mathbf{y}^{(\alpha)}(t), t, \boldsymbol{\lambda}) = \varphi^{(r_\alpha)}(\mathbf{y}^{(0)}(t), t, \boldsymbol{\lambda}) = 0.$$

(ii) and (iii) For $\mathbf{y}_\kappa^{(\alpha)} \in \Omega_\alpha^{\pm\varepsilon}$ at $t_\kappa^- \in [t_{m-\varepsilon}, t_m)$ or $t_\kappa^+ \in (t_m, t_{m+\varepsilon}]$ and $\mathbf{y}_m^{(0)} \in \partial\Omega_{12}$ with $t_m \in (t_{m_1}, t_{m_2})$,

$$(-1)^\alpha \varphi(\mathbf{y}_\kappa^{(\alpha)}, t_\kappa^\pm, \boldsymbol{\lambda}) < 0.$$

Introduce $0 < \varepsilon_1 = |t_{m\pm\varepsilon} - t_\kappa^\pm| < |t_{m\pm\varepsilon} - t_m| = \varepsilon$ for $t_m > t_\kappa^-$ and $t_m < t_\kappa^+$. Because of

$$\begin{aligned} &\varphi(\mathbf{y}_{m\pm\varepsilon_1}^{(\alpha)}, t_{m\pm\varepsilon_1}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}_\kappa^{(\alpha)}, t_\kappa^\pm, \boldsymbol{\lambda}) \\ &= \sum\nolimits_{s_\alpha=1}^{2k_\alpha} \frac{1}{s_\alpha!} \varphi^{(s_\alpha)}(\mathbf{y}_\kappa^{(\alpha)}, t_\kappa^\pm, \boldsymbol{\lambda})(\pm\varepsilon_1)^{s_\alpha} \\ &+ \frac{1}{(2k_\alpha+1)!} \varphi^{(2k_\alpha+1)}(\mathbf{y}_\kappa^{(\alpha)}, t_\kappa^\pm, \boldsymbol{\lambda})(\pm\varepsilon_1)^{2k_\alpha+1} + o((\varepsilon_1)^{2k_\alpha+1}), \end{aligned}$$

and once the $(2k_\alpha + 2)$ and higher order terms drop, one obtains

$$
\begin{aligned}
&\varphi(\mathbf{y}^{(\alpha)}_{m\pm\varepsilon_1}, t_{m\pm\varepsilon_1}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}^{(\alpha)}_{\kappa}, t^{\pm}_{\kappa}, \boldsymbol{\lambda})\\
&= \sum\nolimits_{s_\alpha=1}^{2k_\alpha} \frac{1}{s_\alpha!}\varphi^{(s_\alpha)}(\mathbf{y}^{(\alpha)}_{\kappa}, t^{\pm}_{\kappa}, \boldsymbol{\lambda})(\pm\varepsilon_1)^{s_\alpha}\\
&+ \frac{1}{(2k_\alpha+1)!}\varphi^{(2k_\alpha+1)}(\mathbf{y}^{(\alpha)}_{\kappa}, t^{\pm}_{\kappa}, \boldsymbol{\lambda})(\pm\varepsilon_1)^{2k_\alpha+1},
\end{aligned}
$$

$$
\begin{aligned}
&\lim_{t^{\pm}_{\kappa}\to t_{m\pm}} [\varphi(\mathbf{y}^{(\alpha)}_{m\pm\varepsilon_1}, t_{m\pm\varepsilon_1}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}^{(\alpha)}_{\kappa}, t^{\pm}_{\kappa}, \boldsymbol{\lambda})]\\
&= \sum\nolimits_{s_\alpha=1}^{2k_\alpha} \lim_{t^{\pm}_{\kappa}\to t_{m\pm}} \frac{1}{s_\alpha!}\varphi^{(s_\alpha)}(\mathbf{y}^{(\alpha)}_{\kappa}, t^{\pm}_{\kappa}, \boldsymbol{\lambda})(\pm\varepsilon_1)^{s_\alpha}\\
&+ \lim_{t^{\pm}_{\kappa}\to t_{m\pm}} \frac{1}{(2k_\alpha+1)!}\varphi^{(2k_\alpha+1)}(\mathbf{y}^{(\alpha)}_{\kappa}, t^{\pm}_{\kappa}, \boldsymbol{\lambda})(\pm\varepsilon_1)^{2k_\alpha+1}.
\end{aligned}
$$

Definition 2.21 for $t_m \in (t_{m_1}, t_{m_2})$ with t^-_κ gives

$$
\begin{aligned}
&\lim_{t^{-}_{\kappa}\to t_{m-}} (-1)^{\alpha}\varphi^{(2k_\alpha+1)}(\mathbf{y}^{(\alpha)}_{\kappa}, t^{-}_{\kappa}, \boldsymbol{\lambda}) > 0,\\
&\lim_{t^{-}_{\kappa}\to t_{m}} \varphi^{(2k_\alpha+1)}(\mathbf{y}^{(\alpha)}_{\kappa}, t^{-}_{\kappa}, \boldsymbol{\lambda}) = \varphi^{(2k_\alpha+1)}(\mathbf{y}^{(\alpha)}_{m}, t_m, \boldsymbol{\lambda}) = 0.
\end{aligned}
$$

However, using Eq. (3.57), the condition in Definition 3.13 is obtained. Definition 3.22 for $t_m \notin [t_{m_1}, t_{m_2}]$ with t^+_κ leads to

$$
\begin{aligned}
&\lim_{t^{+}_{\kappa}\to t_{m+}} (-1)^{\alpha}\varphi^{(2k_\alpha+1)}(\mathbf{y}^{(\alpha)}_{\kappa}, t^{+}_{\kappa}, \boldsymbol{\lambda}) > 0,\\
&\lim_{t^{+}_{\kappa}\to t_{m}} \varphi^{(2k_\alpha+1)}(\mathbf{y}^{(\alpha)}_{\kappa}, t^{+}_{\kappa}, \boldsymbol{\lambda}) = \varphi^{(2k_\alpha+1)}(\mathbf{y}^{(\alpha)}_{m}, t_m, \boldsymbol{\lambda}) = 0.
\end{aligned}
$$

However, using Eq. (3.58), the condition in Definition 3.14 is obtained.

(iv) Similarly, for $\mathbf{y}^{(\alpha)}_{\kappa} \in \Omega^{\pm\varepsilon}_{\alpha}$ at time $t^-_\kappa \in [t_{m-\varepsilon}, t_{m-})$ or $t^+_\kappa \in (t_{m+}, t_{m+\varepsilon}]$ and $\mathbf{y}^{(0)}_{m} \in \partial\Omega_{12}$ with $t_m = t_{m_1}$ and t_{m_2},

$$
\begin{aligned}
&\varphi(\mathbf{y}^{(\alpha)}_{m\pm\varepsilon_1}, t_{m\pm\varepsilon_1}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}^{(\alpha)}_{\kappa}, t^{\pm}_{\kappa}, \boldsymbol{\lambda})\\
&= \sum\nolimits_{s_\alpha=1}^{2k_\alpha+1} \frac{1}{s_\alpha!}\varphi^{(s_\alpha)}(\mathbf{y}^{(\alpha)}_{\kappa}, t^{\pm}_{\kappa}, \boldsymbol{\lambda})(\pm\varepsilon_1)^{s_\alpha}\\
&+ \frac{1}{(2k_\alpha+2)!}\varphi^{(2k_\alpha+2)}(\mathbf{y}^{(\alpha)}_{\kappa}, t^{\pm}_{\kappa}, \boldsymbol{\lambda})(\pm\varepsilon_1)^{2k_\alpha+2} + o((\varepsilon_1)^{2k_\alpha+2})
\end{aligned}
$$

Ignoring the $(2k_\alpha + 3)$term or higher order terms, one obtains

$$
\begin{aligned}
&\lim_{t^{\pm}_{\kappa}\to t_{m\pm}} [\varphi(\mathbf{y}^{(\alpha)}_{m\pm\varepsilon_1}, t_{m\pm\varepsilon_1}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}^{(\alpha)}_{\kappa}, t^{\pm}_{\kappa}, \boldsymbol{\lambda})]\\
&= \sum\nolimits_{s_\alpha}^{2k_\alpha+1} \lim_{t^{\pm}_{\kappa}\to t_{m\pm}} \frac{1}{(s_\alpha)!}\varphi^{(s_\alpha)}(\mathbf{y}^{(\alpha)}_{\kappa}, t^{\pm}_{\kappa}, \boldsymbol{\lambda})(\pm\varepsilon_1)^{s_\alpha}\\
&+ \lim_{t^{\pm}_{\kappa}\to t_{m\pm}} \frac{1}{(2k_\alpha+2)!}\varphi^{(2k_\alpha+2)}\varphi^{(2k_\alpha+2)}(\mathbf{y}^{(\alpha)}_{\kappa}, t^{\pm}_{\kappa}, \boldsymbol{\lambda})(\pm\varepsilon_1)^{2k_\alpha+2}
\end{aligned}
$$

Using $\lim_{\kappa_i \to m_i \pm} \varphi^{(s_\alpha)}(\mathbf{y}_\kappa^{(\alpha)}, t_\kappa^\pm, \boldsymbol{\lambda}) = 0$ $(s_\alpha = 1, 2, \cdots 2k_\alpha + 1)$, the foregoing equation gives

$$\begin{aligned}
&\lim_{t_\kappa^\pm \to t_{m\pm}} [\varphi(\mathbf{y}_{m\pm\varepsilon_1}^{(\alpha)}, t_{m\pm\varepsilon_1}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}_\kappa^{(\alpha)}, t_\kappa^\pm, \boldsymbol{\lambda})] \\
&= \lim_{t_\kappa^\pm \to t_{m\pm}} \frac{1}{(2k_\alpha + 2)!} \varphi^{(2k_\alpha+2)}(\mathbf{y}_\kappa^{(\alpha)}, t_\kappa^\pm, \boldsymbol{\lambda})(\pm\varepsilon_1)^{2k_\alpha+2}
\end{aligned}$$

If $\lim_{t_\kappa^\pm \to t_{m\pm}} \varphi^{(2k_\alpha+2)}(\mathbf{y}_\kappa^{(\alpha)}, t_\kappa^\pm, \boldsymbol{\lambda}) < 0$, one obtains

$$\lim_{t_\kappa^\pm \to t_{m\pm}} [\varphi(\mathbf{y}_{m\pm\varepsilon_1}^{(\alpha)}, t_{m\pm\varepsilon_1}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}_\kappa^{(\alpha)}, t_\kappa^\pm, \boldsymbol{\lambda})] < 0$$

From Definition 2.24, the point $(\mathbf{x}_{m_i\pm}^{(\alpha)}, t_{m_i\pm})$ $(i = 1, 2)$ is tangential point to the constraint. The synchronization at such a point appears or disappears. However, from the conditions in Definition 2.24, Eq. (3.59) can be obtained. This theorem is proved. □

Consider the foregoing two theorems with $t_{m_1} \to -\infty$ and $t_{m_2} \to \infty$. For this case, once the two dynamical systems to the constraint are synchronized, such synchronization can keep forever. To explain the two theorems, the synchronization of the flows of two dynamical systems on the boundary $\partial\Omega_{12}$ is in Fig. 3.7. Any point of a constraint flow on the constraint boundary is expressed by $(\mathbf{y}_m^{(0)}, t_m)$ for synchronization. In the two domains, the resultant flows in the vicinity of the constraint boundary are expressed by $(\mathbf{y}_{m-\varepsilon}^{(\alpha)}, t_{m-\varepsilon})$ $(\alpha = 1, 2)$. The onset point is denoted by $(\mathbf{y}_{m_1}^{(0)}, t_{m_1})$. For $t_m > t_{m_1}$ and $t_{m_2} \to \infty$, all the flows of the resultant system of two dynamical systems will be on the constraint boundary. Thus, the synchronization of the two dynamical systems to the constraint is an absolute synchronization. The starting point of a resultant flow for the synchronization can occur at any time $t_m > t_{m_1}$. However, if t_{m_2} is finite, the two dynamical systems to the constraint can be synchronized only in a finite time interval of $t \in (t_{m_1}, t_{m_2})$. To the point on the boundary at time $t = t_{m_2}$, such synchronization will disappear. Further, the resultant flow on the constraint boundary for synchronization vanishing will enter into the domain, which cannot be synchronized any more in sense of Eq. (3.3). Such synchronization is very easily realized through the discontinuous vector fields to the two dynamical systems to the constraint boundary. For the synchronization of slave and master systems to the constraint, a slave system is controlled by discontinuous, external vector fields in order to make it synchronize with the master system.

For $\mathbb{F}^{(\alpha)}(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\pi}^{(\alpha)}) = \mathbb{F}^{(0)}(\mathbf{y}^{(0)}, t, \boldsymbol{\lambda})$ at $\mathbf{y}^{(\alpha)} = \mathbf{y}^{(0)}$ $(\alpha \in \{1, 2\})$, the synchronization of two dynamical systems to a specific constraint requires $D^k\varphi(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\lambda}) = D^k\varphi(\mathbf{y}^{(0)}, t, \boldsymbol{\lambda}) = 0$. If a resultant system of two different dynamical systems is continuous to the constraint boundary, it is very difficult to make the two different dynamical systems be synchronized with a specific constraint. Most of such synchronization is *asymptotic* as $t \to \infty$. To make the synchronization of two dynamical systems to a specific constraint possible, one often considers control

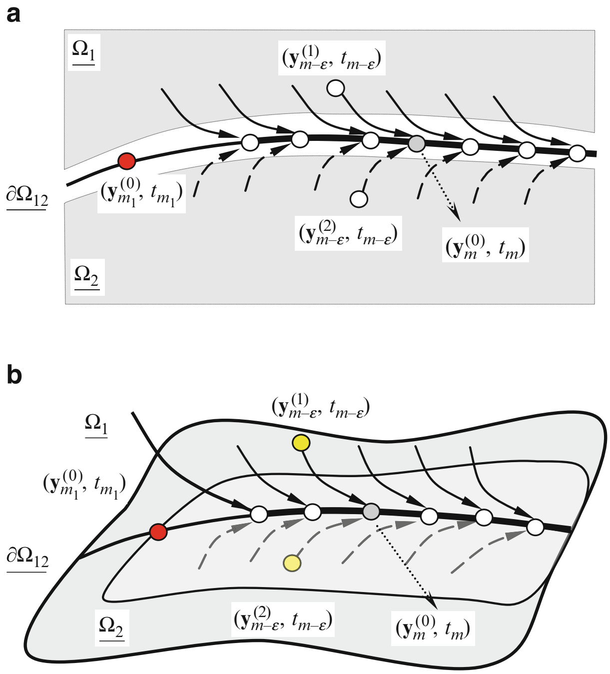

Fig. 3.7 (**a**) A cross-section view and (**b**) a three-dimensional view for an absolute synchronization of two dynamical systems to the constraint in vicinity of the constraint boundary $\partial\Omega_{12}$ in $(n_r + n_s)$-dimensional state space. Any point for synchronization on the constraint boundary is expressed by $(\mathbf{y}^{(0)}_m, t_m)$. In two domains, the resultant flows in the vicinity of the constraint boundary are expressed by $(\mathbf{y}^{(\alpha)}_{m-\varepsilon}, t_{m-\varepsilon})$ $(\alpha = 1, 2)$. The onset point on the constraint boundary is $(\mathbf{y}^{(0)}_{m_1}, t_{m_1})$, depicted by a *red circular symbol*

schemes to realize the synchronization via adjusting vector fields. Next, consider the resultant system of two different dynamical systems to be discontinuous to the constraint boundary.

For $\mathbb{F}^{(\alpha)}(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\pi}^{(\alpha)}) \neq \mathbb{F}^{(0)}(\mathbf{y}^{(0)}, t, \boldsymbol{\lambda})$ at $\mathbf{y}^{(\alpha)} = \mathbf{y}^{(0)}$ $(\alpha \in \{1, 2\})$, the synchronization of two dynamical systems with a specific constraint satisfies

$$\frac{d^k}{dt^k}\varphi(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\lambda}) \neq \frac{d^k}{dt^k}\varphi(\mathbf{y}^{(0)}, t, \boldsymbol{\lambda}) = 0 \quad \text{for } k = 1, 2, \cdots. \tag{3.60}$$

To distinguish $\mathbf{y}^{(\alpha)}_{s-}$ from $\mathbf{y}^{(0)}_s$ at time $t_s \in [t_m, t_{m+1}]$, a point $\mathbf{y}^{(\alpha)}_{s-} \in \Omega^{-\varepsilon}_{\alpha}$ in the domain infinitesimally approaches a point $\mathbf{y}^{(0)}_s \in \partial\Omega_{12}$ on the constraint boundary at time t. For $\mathbf{y}^{(\alpha)}_{s-} \in \Omega^{-\varepsilon}_{\alpha}$ (or $\mathbf{y}^{(\alpha)}_{s-} \notin \partial\Omega_{12}$), the corresponding differentiation of vector fields with respect to state variables can be carried out. For $\mathbf{y}^{(0)}_s \in \partial\Omega_{12}$ on the constraint boundary, such differentiation cannot be done for $t' \in (t_s - \varepsilon, t_s)$ (any small $\varepsilon > 0$) because the vector fields $(\mathbb{F}^{(\alpha)}(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\pi}^{(\alpha)})$, $\alpha \in \{1, 2\})$ to the constraint boundary $\partial\Omega_{12}$ are discontinuous (i.e., $\mathbb{F}^{(0)}(\mathbf{y}^{(0)}_s, t_s, \boldsymbol{\lambda}) \neq \mathbb{F}^{(\alpha)}(\mathbf{y}^{(\alpha)}_{s-}, t_{s-}, \boldsymbol{\pi}^{(\alpha)})$

for $\mathbf{y}_{s-}^{(\alpha)} = \mathbf{y}_s^{(0)}$ at time $t_s = t_{s-}$). Therefore, the time t_s will be replaced by $t_{s-} = t_s - 0$ for a point $\mathbf{y}_{s-}^{(\alpha)} \in \Omega_\alpha$. Under the constraint condition in Eq. (3.3), the corresponding theorem is presented for the synchronization of two dynamical systems in Eqs. (3.1) and (3.2) as follows.

Theorem 3.3 *Consider two dynamical systems in Eqs. (3.1) and (3.2) with constraint in Eq. (3.3). For $\mathbf{y}_{m\pm}^{(\alpha)} \in \Omega_\alpha$ $(\alpha \in \{1,2\})$ and $\mathbf{y}_m^{(0)} \in \partial\Omega_{12}$ at time t_m, $\mathbf{y}_{m\pm}^{(\alpha)} = \mathbf{y}_m^{(0)}$. For any small $\varepsilon > 0$, there is a time interval $[t_{m-\varepsilon}, t_{m+\varepsilon}]$. At $\mathbf{y}^{(\alpha)} \in \Omega_\alpha^{\pm\varepsilon}$ for time $t \in [t_{m-\varepsilon}, t_{m+\varepsilon}]$, the constraint function $\varphi(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\lambda})$ is C^{r_α}-continuous ($r_\alpha \geq 3$) and $|\varphi^{(r_\alpha+1)}(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\lambda})| < \infty$. For $\mathbf{y}^{(\alpha)} \in \Omega_\alpha$ and $\mathbf{y}^{(0)} \in \partial\Omega_{12}$, suppose $\mathbb{F}^{(\alpha)}(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\pi}^{(\alpha)}) \neq \mathbb{F}^{(0)}(\mathbf{y}^{(0)}, t, \boldsymbol{\lambda})$ for $\mathbf{y}^{(\alpha)} = \mathbf{y}^{(0)}$. The two dynamical systems in Eqs. (3.1) and (3.2) to the constraint in Eq. (3.3) are synchronized for time $t \in [t_{m_1}, t_{m_2}]$ if and only if*

(i) *for $\mathbf{y}_{m\pm}^{(\alpha)} = \mathbf{y}_m^{(0)}$ and $\mathbf{y}^{(\alpha)} \in \Omega_\alpha$ ($\alpha \in \{1,2\}$) at time $t = t_m \in [t_{m_1}, t_{m_2}]$*

$$\varphi(\mathbf{y}_{m\pm}^{(\alpha)}, t_{m\pm}, \boldsymbol{\lambda}) = \varphi(\mathbf{y}_m^{(0)}, t_m, \boldsymbol{\lambda}) = 0 \tag{3.61}$$

(ii) *for time $t_m \in (t_{m_1}, t_{m_2})$,*

$$\mathbf{y}_{m-}^{(\alpha)} = \mathbf{y}_m^{(0)} \text{ and } (-1)^\alpha \varphi^{(1)}(\mathbf{y}_{m-}^{(\alpha)}, t_{m-}, \boldsymbol{\lambda}) > 0 \text{ for } \alpha = 1, 2 \tag{3.62}$$

(iii) *with penetration at time $t = t_{m_i}$, $\mathbf{y}_{m_i\pm}^{(\alpha)} = \mathbf{y}_{m_i}^{(0)}$ ($i = 1, 2$)*

$$\begin{aligned} &\varphi^{(1)}(\mathbf{y}_{m_i\pm}^{(\alpha)}, t_{m_i\pm}, \boldsymbol{\lambda}) = 0 \text{ and } (-1)^\alpha \varphi^{(2)}(\mathbf{y}_{m_i\pm}^{(\alpha)}, t_{m_i\pm}, \boldsymbol{\lambda}) < 0, \\ &(-1)^\alpha \varphi^{(1)}(\mathbf{y}_{m_i-}^{(\beta)}, t_{m_i-}, \boldsymbol{\lambda}) > 0 \quad \text{for } \alpha,\beta \in \{1,2\} \text{ and } \beta \neq \alpha \end{aligned} \tag{3.63}$$

or with desynchronization at time $t = t_{m_i}$, $\mathbf{y}_{m_i\pm}^{(\alpha)} = \mathbf{y}_{m_i}^{(0)}$ ($i = 1, 2$)

$$\begin{aligned} &\varphi^{(1)}(\mathbf{y}_{m_i\pm}^{(\alpha)}, t_{m_i\pm}, \boldsymbol{\lambda}) = 0 \text{ and } (-1)^\alpha \varphi^{(2)}(\mathbf{y}_{m_i\pm}^{(\alpha)}, t_{m_i\pm}, \boldsymbol{\lambda}) < 0, \\ &\varphi^{(1)}(\mathbf{y}_{m_i\pm}^{(\beta)}, t_{m_i\pm}, \boldsymbol{\lambda}) = 0 \text{ and } (-1)^\beta \varphi^{(2)}(\mathbf{y}_{m_i\pm}^{(\beta)}, t_{m_i\pm}, \boldsymbol{\lambda}) < 0 \\ &\text{for } \alpha,\beta \in \{1,2\} \text{ and } \beta \neq \alpha. \end{aligned} \tag{3.64}$$

Proof (i) Consider two dynamical systems in Eqs. (3.1) and (3.2) with a constraint condition in Eq. (3.3). From Definition 3.10, the constraint functions for the constraint boundary $\partial\Omega_{12}$ and domains Ω_α ($\alpha = 1, 2$) are given by

$$\begin{aligned} &\varphi(\mathbf{y}^{(0)}, t, \boldsymbol{\lambda}) = 0 \quad \text{for } \mathbf{y}^{(0)} \in \partial\Omega_{12}, \\ &(-1)^\alpha \varphi(\mathbf{y}^{(\alpha)}, t, \lambda) < 0 \quad \text{for } \mathbf{y}^{(\alpha)} \in \Omega_\alpha, \quad \alpha = 1, 2. \end{aligned}$$

For $t = t_{m-}$ and $\mathbf{y}^{(\alpha)} = \mathbf{y}_{m-}^{(\alpha)} \in \Omega_\alpha$ ($\alpha \in \{1, 2\}$), we have $\mathbf{y}_{m-}^{(\alpha)} = \mathbf{y}_m^{(0)} \in \partial\Omega_{12}$. Further,

$$\varphi(\mathbf{y}_{m-}^{(\alpha)}, t_{m-}, \boldsymbol{\lambda}) = \varphi(\mathbf{y}_m^{(0)}, t_m, \boldsymbol{\lambda}) = 0.$$

Equation (3.61) is obtained, vice versa. Because $\mathbb{F}^{(\alpha)}(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\pi}^{(\alpha)}) \neq \mathbb{F}^{(0)}(\mathbf{y}^{(0)}, t, \boldsymbol{\lambda})$ on the constraint boundary $\partial\Omega_{12}$, one obtains $d^{r_\alpha}\mathbf{y}^{(\alpha)}/dt^{r_\alpha} \neq d^{r_\alpha}\mathbf{y}^{(0)}/dt^{r_\alpha}$ for all time t. Thus, the following equation cannot always hold for all $r_\alpha = 1, 2, \cdots$

$$\varphi^{(r_\alpha)}(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\lambda}) \neq \varphi^{(r_\alpha)}(\mathbf{y}^{(0)}, t, \boldsymbol{\lambda}) = 0.$$

(ii) For time $t_m \in (t_{m_1}, t_{m_2})$, $\mathbf{y}^{(\alpha)}_{m-} = \mathbf{y}^{(0)}_m \in \partial\Omega_{12}$. Consider a point $\mathbf{y}^{(\alpha)}_{m-\varepsilon} \in \Omega^{\varepsilon}_\alpha$ for $t_{m-\varepsilon} = t_m - \varepsilon$ in the neighborhood of $\mathbf{y}^{(0)}_m \in \partial\Omega_{12}$ and $\varepsilon > 0$. We have

$$\varphi(\mathbf{y}^{(\alpha)}_{m-\varepsilon}, t_{m-\varepsilon}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}^{(\alpha)}_{m-}, t_{m-}, \boldsymbol{\lambda}) = -\varphi^{(1)}(\mathbf{y}^{(\alpha)}_{m-}, t_{m-}, \boldsymbol{\lambda})\varepsilon + o(\varepsilon).$$

Because of any selection of $\varepsilon > 0$, if

$$(-1)^\alpha \varphi^{(1)}(\mathbf{y}^{(\alpha)}_{m-}, t_{m-}, \boldsymbol{\lambda}) > 0 \text{ for } \alpha = 1, 2$$

then

$$(-1)^\alpha [\varphi(\mathbf{y}^{(\alpha)}_{m-\varepsilon}, t_{m-\varepsilon}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}^{(\alpha)}_{m-}, t_{m-}, \boldsymbol{\lambda})] < 0.$$

From Definition 3.15, the two dynamical systems to a specific constraint are synchronized for time interval of $t_m \in (t_{m_1}, t_{m_2})$. However, if the foregoing equation is satisfied, Eq. (3.62) is achieved.

(iii) At time $t = t_{m_i}$, $\mathbf{y}^{(\alpha)}_{m_i\pm} = \mathbf{y}^{(0)}_{m_i} \in \partial\Omega_{12}$. Consider a point $\mathbf{y}^{(\alpha)}_{m_i\pm\varepsilon} \in \Omega_\alpha$ $(\alpha = 1, 2)$ for $t_{m_i\pm\varepsilon} = t_{m_i} \pm \varepsilon$ in the neighborhood of $\mathbf{y}^{(0)}_m \in \partial\Omega_{12}$ and $\varepsilon > 0$. The Taylor series expansion gives

$$\begin{aligned}&\varphi(\mathbf{y}^{(\alpha)}_{m_2\pm\varepsilon}, t_{m_2\pm\varepsilon}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}^{(\alpha)}_{m_i\pm}, t_{m_i\pm}, \boldsymbol{\lambda})\\ &= \pm\varphi^{(1)}(\mathbf{y}^{(\alpha)}_{m_i\pm}, t_{m_i\pm}, \boldsymbol{\lambda})\varepsilon + \frac{1}{2!}\varphi^{(2)}(\mathbf{y}^{(\alpha)}_{m_i\pm}, t_{m_i\pm}, \boldsymbol{\lambda})\varepsilon^2 + o(\varepsilon^2)\end{aligned}$$

If the third and higher order terms are dropped in the foregoing equation in Ω_α $(\alpha = 1, 2)$, with the condition

$$\varphi^{(1)}(\mathbf{y}^{(\alpha)}_{m_i\pm}, t_{m_i\pm}, \boldsymbol{\lambda}) = 0$$

the following equation is achieved.

$$\varphi(\mathbf{y}^{(\alpha)}_{m_2\pm\varepsilon}, t_{m_2\pm\varepsilon}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}^{(\alpha)}_{m_i\pm}, t_{m_i\pm}, \boldsymbol{\lambda}) = \frac{1}{2!}\varphi^{(2)}(\mathbf{y}^{(\alpha)}_{m_i\pm}, t_{m_i\pm}, \boldsymbol{\lambda})\varepsilon^2.$$

If $\varphi^{(1)}(\mathbf{y}^{(\alpha)}_{m_i\pm}, t_{m_i\pm}, \boldsymbol{\lambda}) \neq 0$ and only the first-order term in the Taylor series expansion is considered, one gets

$$\varphi(\mathbf{y}^{(\alpha)}_{m_2\pm\varepsilon}, t_{m_2\pm\varepsilon}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}^{(\alpha)}_{m_i\pm}, t_{m_i\pm}, \boldsymbol{\lambda}) = \pm\varphi^{(1)}(\mathbf{y}^{(\alpha)}_{m_i\pm}, t_{m_i\pm}, \boldsymbol{\lambda})\varepsilon$$

For $\alpha, \beta \in \{1,2\}$ and $\alpha \neq \beta$, from Definition 3.19, the disappearance and appearance of synchronization with the penetration require

$$\begin{aligned}(-1)^{\alpha}[\varphi(\mathbf{y}^{(\alpha)}_{m_2\pm\varepsilon}, t_{m_2\pm\varepsilon}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}^{(\alpha)}_{m_i\pm}, t_{m_i\pm}, \boldsymbol{\lambda})] < 0,\\ (-1)^{\beta}[\varphi(\mathbf{y}^{(\beta)}_{m_2-\varepsilon}, t_{m_2-\varepsilon}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}^{(\beta)}_{m_i-}, t_{m_i-}, \boldsymbol{\lambda})] < 0,\end{aligned}$$

from which Eq. (3.63) is obtained, vice versa.

(iv) For $\alpha, \beta \in \{1,2\}$ and $\alpha \neq \beta$, from Definition 18, the disappearance and onset of synchronization with the desynchronization require

$$\begin{aligned}(-1)^{\alpha}[\varphi(\mathbf{y}^{(\alpha)}_{m_2\pm\varepsilon}, t_{m_2\pm\varepsilon}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}^{(\alpha)}_{m_i\pm}, t_{m_i\pm}, \boldsymbol{\lambda})] < 0,\\ (-1)^{\beta}[\varphi(\mathbf{y}^{(\beta)}_{m_2\pm\varepsilon}, t_{m_2\pm\varepsilon}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}^{(\beta)}_{m_i\pm}, t_{m_i\pm}, \boldsymbol{\lambda})] < 0,\end{aligned}$$

from which Eq. (3.64) is obtained, vice versa. Therefore, this theorem is proved. □

From the foregoing theorem, the synchronization of two dynamical systems to a special constraint requires that the first-order derivative of the constraint function be less than zero. The *onset and vanishing* conditions of the synchronization in Eqs. (3.61) and (3.62) are the *vanishing and onset* conditions relative to the penetration and desynchronization, respectively. If the first-order derivative is zero, under what conditions can two dynamical systems to a special constraint be synchronized together in sense of Eq. (3.3)? The following theorem will consider the synchronization of two dynamical systems to a special constraint with higher order singularity.

Theorem 3.4 *Consider two dynamical systems in Eqs. (3.1) and (3.2) with constraint in Eq. (3.3). For* $\mathbf{y}^{(\alpha)}_{m\pm} \in \boldsymbol{\Omega}_{\alpha}$ $(\alpha \in \{1,2\})$ *and* $\mathbf{y}^{(0)}_{m} \in \partial\boldsymbol{\Omega}_{12}$ *at time* t_m, $\mathbf{y}^{(\alpha)}_{m\pm} = \mathbf{y}^{(0)}_{m}$. *For any small* $\varepsilon > 0$, *there is a time interval* $[t_{m-\varepsilon}, t_{m+\varepsilon}]$. *At* $\mathbf{y}^{(\alpha)} \in \boldsymbol{\Omega}^{\pm\varepsilon}_{\alpha}$ *for time* $t \in [t_{m-\varepsilon}, t_{m+\varepsilon}]$, *the constraint function* $\varphi(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\lambda})$ *is* C^{r_α}*-continuous* $(r_\alpha \geq 2k_\alpha + 1)$ *and* $|\varphi^{(r_\alpha+1)}(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\lambda})| < \infty$. *For* $\mathbf{y}^{(\alpha)} \in \boldsymbol{\Omega}_{\alpha}$ *and* $\mathbf{y}^{(0)} \in \partial\boldsymbol{\Omega}_{12}$, *suppose* $\mathbb{F}^{(\alpha)}(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\pi}^{(\alpha)}) \neq \mathbb{F}^{(0)}(\mathbf{y}^{(0)}, t, \boldsymbol{\lambda})$ *for* $\mathbf{y}^{(\alpha)} = \mathbf{y}^{(0)}$. *The two dynamical systems in Eqs. (3.1) and (3.2) to the constraint in Eq. (3.3) are synchronized of the* $(2k_\alpha : 2k_\beta)$*-type for time* $t \in [t_{m_1}, t_{m_2}]$ *if and only if*

(i) *for* $\mathbf{y}^{(\alpha)}_{m\pm} = \mathbf{y}^{(0)}_{m}$ *and* $\mathbf{y}^{(\alpha)}(t) \in \boldsymbol{\Omega}_{\alpha}$ *(*$\alpha \in \{1,2\}$*) at time* $t = t_m \in [t_{m_1}, t_{m_2}]$

$$\varphi(\mathbf{y}^{(\alpha)}_{m\pm}, t_{m\pm}, \boldsymbol{\lambda}) = \varphi(\mathbf{y}^{(0)}_{m}, t_m, \boldsymbol{\lambda}) = 0 \tag{3.65}$$

(ii) *for time* $t_m \in (t_{m_1}, t_{m_2})$,

$$\begin{aligned}&\mathbf{y}^{(\alpha)}_{m-} = \mathbf{y}^{(0)}_{m} \text{ and } \varphi^{(s_\alpha)}(\mathbf{y}^{(\alpha)}_{m-}, t_{m-}, \boldsymbol{\lambda}) = 0 \quad \text{for } s_\alpha = 1, 2, \cdots, 2k_\alpha,\\ &(-1)^{\alpha}\varphi^{(2k_\alpha+1)}(\mathbf{y}^{(\alpha)}_{m-}, t_{m-}, \boldsymbol{\lambda}) > 0 \quad \text{for } \alpha = 1, 2.\end{aligned} \tag{3.66}$$

(iii) *with the* $(2k_\alpha : 2k_\beta)$*-penetration for time* $t = t_{m_i}$, $\mathbf{y}_{m_i}^{(\alpha)} = \mathbf{y}_{m_i}^{(0)}$ *(*$i = 1, 2$*),*

$$\begin{aligned}
&\varphi^{(s_\alpha)}(\mathbf{y}_{m_i\pm}^{(\alpha)}, t_{m_i\pm}, \boldsymbol{\lambda}) = 0 \quad (s_\alpha = 1, 2, \cdots, 2k_\alpha + 1),\\
&(-1)^\alpha \varphi^{(2k_\alpha+2)}(\mathbf{y}_{m-}^{(\alpha)}, t_{m-}, \boldsymbol{\lambda}) < 0,\\
&\varphi^{(s_\beta)}(\mathbf{y}_{m_i-}^{(\beta)}, t_{m_i-}, \boldsymbol{\lambda}) = 0 \quad (s_\beta = 1, 2, \cdots 2k_\beta),\\
&(-1)^\beta \varphi^{(2k_\beta+1)}(\mathbf{y}_{m_i-}^{(\beta)}, t_{m_i-}, \boldsymbol{\lambda}) > 0 \quad \text{for } \alpha, \beta \in \{1,2\} \text{ and } \beta \neq \alpha.
\end{aligned} \tag{3.67}$$

or with the $(2k_\alpha : 2k_\beta)$*-desynchronization for time* $t = t_{m_i}$, $\mathbf{y}_{m_i}^{(\alpha)} = \mathbf{y}_{m_i}^{(0)}$ *(*$i = 1, 2$*),*

$$\begin{aligned}
&\varphi^{(s_\alpha)}(\mathbf{y}_{m_i\pm}^{(\alpha)}, t_{m_i\pm}, \boldsymbol{\lambda}) = 0 \quad (s_\alpha = 1, 2, \cdots, 2k_\alpha + 1),\\
&(-1)^\alpha \varphi^{(2k_\alpha+2)}(\mathbf{y}_{m-}^{(\alpha)}, t_{m-}, \boldsymbol{\lambda}) < 0,\\
&\varphi^{(s_\beta)}(\mathbf{y}_{m_i-}^{(\beta)}, t_{m_i-}, \boldsymbol{\lambda}) = 0 \quad (s_\beta = 1, 2, \cdots 2k_\beta + 1),\\
&(-1)^\beta \varphi^{(2k_\beta+2)}(\mathbf{y}_{m_i-}^{(\beta)}, t_{m_i-}, \boldsymbol{\lambda}) < 0 \quad \text{for } \alpha, \beta \in \{1,2\} \text{ and } \beta \neq \alpha.
\end{aligned} \tag{3.68}$$

Proof Consider two dynamical systems in Eqs. (3.1) and (3.2) with a constraint condition in Eq. (3.3).

(i) From Definition 3.10, the constraint functions for the constraint boundary $\partial\Omega_{12}$ and domains Ω_α ($\alpha = 1, 2$) are given by

$$\begin{aligned}
&\varphi(\mathbf{y}^{(0)}, t, \boldsymbol{\lambda}) = 0 \text{ for } \mathbf{y}^{(0)} \in \partial\Omega_{12},\\
&(-1)^\alpha \varphi(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\lambda}) < 0 \text{ for } \mathbf{y}^{(\alpha)} \in \Omega_\alpha,\ \alpha = 1, 2.
\end{aligned}$$

For $t = t_m \in [t_{m_1}, t_{m_2}]$ and $\mathbf{y}^{(\alpha)} = \mathbf{y}_m^{(\alpha)} \in \Omega_\alpha$ ($\alpha \in \{1,2\}$), we have $\mathbf{y}_{m-}^{(\alpha)} = \mathbf{y}_m^{(0)} \in \partial\Omega_{12}$. Further,

$$\varphi(\mathbf{y}_{m-}^{(\alpha)}, t_{m-}, \boldsymbol{\lambda}) = \varphi(\mathbf{y}_m^{(0)}, t_m, \boldsymbol{\lambda}) = 0,$$

Equation (3.65) is obtained, vice versa. Because $\mathbb{F}^{(\alpha)}(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\pi}^{(\alpha)}) \neq \mathbb{F}^{(0)}(\mathbf{y}^{(0)}, t, \boldsymbol{\lambda})$ on the constraint boundary $\partial\Omega_{12}$, one obtains $d^{r_\alpha}\mathbf{y}^{(\alpha)}/dt^{r_\alpha} \neq d^{r_\alpha}\mathbf{y}^{(0)}/dt^{r_\alpha}$ for all time t. Thus, the following equation cannot always hold for all $r_\alpha = 1, 2, \cdots$

$$\varphi^{(r_\alpha)}(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\lambda}) \neq \varphi^{(r_\alpha)}(\mathbf{y}^{(0)}, t, \boldsymbol{\lambda}) = 0.$$

(ii) For time $t_m \in (t_{m_1}, t_{m_2})$, $\mathbf{y}_{m-}^{(\alpha)} = \mathbf{y}_m^{(0)} \in \partial\Omega_{12}$. Consider a point $\mathbf{y}_{m-\varepsilon}^{(\alpha)} \in \Omega_\alpha^\varepsilon$ for $t_{m-\varepsilon} = t_m - \varepsilon$ in the neighborhood of $\mathbf{y}_m^{(0)} \in \partial\Omega_{12}$ and $\varepsilon > 0$. The following Taylor series expansion is achieved.

$$\begin{aligned}
&\varphi(\mathbf{y}_{m-\varepsilon}^{(\alpha)}, t_{m-\varepsilon}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}_{m-}^{(\alpha)}, t_{m-}, \boldsymbol{\lambda}) = \sum\nolimits_{s_\alpha=1}^{2k_\alpha} \frac{1}{s_\alpha!} \varphi^{(s_\alpha)}(\mathbf{y}_{m-}^{(\alpha)}, t_{m-}, \boldsymbol{\lambda})(-\varepsilon)^{s_\alpha}\\
&- \frac{1}{(2k_\alpha + 1)!} \varphi^{(2k_\alpha+1)}(\mathbf{y}_{m-}^{(\alpha)}, t_{m-}, \boldsymbol{\lambda})\varepsilon^{2k_\alpha+1} + o(\varepsilon^{2k_\alpha+1}).
\end{aligned}$$

Due to the higher order singularity, i.e.,

$$\varphi^{(s_\alpha)}(\mathbf{y}_{m-}^{(\alpha)}, t_{m-}, \boldsymbol{\lambda}) = 0 \quad \text{for } s_\alpha = 1, 2, \cdots, 2k_\alpha$$

and by ignoring of the $(2k_\alpha + 2)$-order and higher order terms, the Taylor series expansion gives

$$\varphi(\mathbf{y}_{m-\varepsilon}^{(\alpha)}, t_{m-\varepsilon}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}_{m-}^{(\alpha)}, t_{m-}, \boldsymbol{\lambda}) = -\frac{1}{(2k_\alpha + 1)!}\varphi^{(2k_\alpha+1)}(\mathbf{y}_{m-}^{(\alpha)}, t_{m-}, \boldsymbol{\lambda})\varepsilon^{2k_\alpha+1}.$$

From Definition 3.22, the synchronization of two dynamical systems to a specific constraint for time $t_m \in (t_{m_1}, t_{m_2})$ requires

$$(-1)^\alpha[\varphi(\mathbf{y}_{m-\varepsilon}^{(\alpha)}, t_{m-\varepsilon}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}_{m-}^{(\alpha)}, t_{m-}, \boldsymbol{\lambda})] < 0.$$

Thus,

$$(-1)^\alpha \varphi^{(2k_\alpha+1)}(\mathbf{y}_{m-}^{(\alpha)}, t_{m-}, \boldsymbol{\lambda}) > 0.$$

However, if $(-1)^\alpha \varphi^{(2k_\alpha+1)}(\mathbf{y}_{m-}^{(\alpha)}, t_{m-}, \boldsymbol{\lambda}) > 0$,

$$(-1)^\alpha[\varphi(\mathbf{y}_{m-\varepsilon}^{(\alpha)}, t_{m-\varepsilon}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}_{m-}^{(\alpha)}, t_{m-}, \boldsymbol{\lambda})] < 0.$$

is achieved, which implies the two dynamical systems to the specific constraint are synchronized for time $t_m \in (t_{m_1}, t_{m_2})$.

(iii) At time $t = t_{m_i}$, $\mathbf{y}_{m_i\pm}^{(\alpha)} = \mathbf{y}_{m_i}^{(0)} \in \partial\Omega_{12}$. Consider a point $\mathbf{y}_{m_i\pm\varepsilon}^{(\alpha)} \in \Omega_\alpha$ for $t_{m_i\pm\varepsilon} = t_{m_i} \pm \varepsilon$ in the neighborhood of $\mathbf{y}_{m_i}^{(0)} \in \partial\Omega_{12}$ and $\varepsilon > 0$. The Taylor series expansion gives

$$\varphi(\mathbf{y}_{m_i\pm\varepsilon}^{(\alpha)}, t_{m_i\pm\varepsilon}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}_{m_i\pm}^{(\alpha)}, t_{m_i\pm}, \boldsymbol{\lambda}) = \sum\nolimits_{s_\alpha=1}^{2k_\alpha+1} \frac{1}{s_\alpha!}\varphi^{(s_\alpha)}(\mathbf{y}_{m_i\pm}^{(\alpha)}, t_{m_i\pm}, \boldsymbol{\lambda})(\pm\varepsilon)^{s_\alpha}$$
$$+ \frac{1}{(2k_\alpha + 2)!}\varphi^{(2k_\alpha+2)}(\mathbf{y}_{m_i\pm}^{(\alpha)}, t_{m_i\pm}, \boldsymbol{\lambda})\varepsilon^{2k_\alpha+2} + o(\varepsilon^{2k_\alpha+2})$$

Because of the higher order singularity of the constraint function in domain Ω_α, i.e.,

$$\varphi^{(s_\alpha)}(\mathbf{y}_{m_i\pm}^{(\alpha)}, t_{m_i\pm}, \boldsymbol{\lambda}) = 0 \quad \text{for } s_\alpha = 1, 2, \cdots, 2k_\alpha$$

and once the higher order terms of $\varepsilon^{2k_\alpha+1}$ are dropped, one obtains

$$\varphi(\mathbf{y}_{m_i\pm\varepsilon}^{(\alpha)}, t_{m_i\pm\varepsilon}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}_{m_i\pm}^{(\alpha)}, t_{m_i\pm}, \boldsymbol{\lambda}) = \pm\frac{1}{(2k_\alpha + 1)!}\varphi^{(2k_\alpha+1)}(\mathbf{y}_{m_i\pm}^{(\alpha)}, t_{m_i\pm}, \boldsymbol{\lambda})\varepsilon^{2k_\alpha+1}.$$

If the following equation exists

$$\varphi^{(s_\alpha)}(\mathbf{y}_{m_i\pm}^{(\alpha)}, t_{m_i\pm}, \boldsymbol{\lambda}) = 0 \quad \text{for } s_\alpha = 1, 2, \cdots, 2k_\alpha + 1$$

and the higher order term of $\varepsilon^{2k_\alpha+2}$ will not be considered, the Taylor series expansion gives

$$\varphi(\mathbf{y}_{m_i\pm\varepsilon}^{(\alpha)}, t_{m_i\pm\varepsilon}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}_{m_i\pm}^{(\alpha)}, t_{m_i\pm}, \boldsymbol{\lambda}) = \frac{1}{(2k_\alpha+2)!}\varphi^{(2k_\alpha+2)}(\mathbf{y}_{m_i\pm}^{(\alpha)}, t_{m_i\pm}, \boldsymbol{\lambda})\varepsilon^{2k_\alpha+2}.$$

From Definition 2.25, the onset and vanishing conditions of the $(2k_\alpha : 2k_\beta)$-synchronization of the two dynamical systems with a corresponding penetration on the constraint boundary $\partial\Omega_{\alpha\beta}$ are

$$\begin{aligned}
&(-1)^\alpha[\varphi(\mathbf{y}_{m_i\mp\varepsilon}^{(\alpha)}, t_{m_i\mp\varepsilon}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}_{m_i\mp}^{(\alpha)}, t_{m_i\mp}, \boldsymbol{\lambda})] < 0,\\
&(-1)^\beta[\varphi(\mathbf{y}_{m_i-\varepsilon}^{(\beta)}, t_{m_i-\varepsilon}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}_{m_i-}^{(\beta)}, t_{m_i-}, \boldsymbol{\lambda})] < 0,
\end{aligned}$$

with

$$\begin{aligned}
&\varphi^{(s_\alpha)}(\mathbf{y}_{m_i\mp}^{(\alpha)}, t_{m_i\mp}, \boldsymbol{\lambda}) = 0 \quad (s_\alpha = 1, 2,\cdots 2k_\alpha + 1),\\
&\varphi^{(s_\beta)}(\mathbf{y}_{m_i-}^{(\beta)}, t_{m_i-}, \boldsymbol{\lambda}) = 0 \quad (s_\beta = 1, 2,\cdots 2k_\beta).
\end{aligned}$$

Thus, one gets

$$(-1)^\alpha\varphi^{(2k_\alpha+2)}(\mathbf{y}_{m_i\mp\varepsilon}^{(\alpha)}, t_{m_i\mp\varepsilon}, \boldsymbol{\lambda}) < 0 \text{ and } (-1)^\beta\varphi^{(2k_\beta+1)}(\mathbf{y}_{m_i-}^{(\beta)}, t_{m_i-}, \boldsymbol{\lambda}) > 0.$$

In other words, Eq. (3.67) is obtained. If Eq. (3.67) holds, the conditions in Definition 3.25 can be obtained for the onset and vanishing condition for synchronization from the penetration.

If the $(2k_\alpha : 2k_\beta)$-synchronization of two dynamical systems to a specific constraint vanishes and appears with a $(2k_\alpha : 2k_\beta)$-desynchronization, the following conditions are required

$$\begin{aligned}
&(-1)^\alpha[\varphi(\mathbf{y}_{m_i\mp\varepsilon}^{(\alpha)}, t_{m_i\mp\varepsilon}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}_{m_i\mp}^{(\alpha)}, t_{m_i\mp}, \boldsymbol{\lambda})] < 0,\\
&(-1)^\beta[\varphi(\mathbf{y}_{m_i\mp\varepsilon}^{(\beta)}, t_{m_i\mp\varepsilon}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}_{m_i\mp}^{(\beta)}, t_{m_i\mp}, \boldsymbol{\lambda})] < 0,
\end{aligned}$$

with the singularity conditions

$$\begin{aligned}
&\varphi^{(s_\alpha)}(\mathbf{y}_{m_i\mp}^{(\alpha)}, t_{m_i\mp}, \boldsymbol{\lambda}) = 0 \quad (s_\alpha = 1, 2,\cdots 2k_\alpha + 1),\\
&\varphi^{(s_\beta)}(\mathbf{y}_{m_i\mp}^{(\beta)}, t_{m_i\mp}, \boldsymbol{\lambda}) = 0 \quad (s_\beta = 1, 2,\cdots 2k_\beta + 1).
\end{aligned}$$

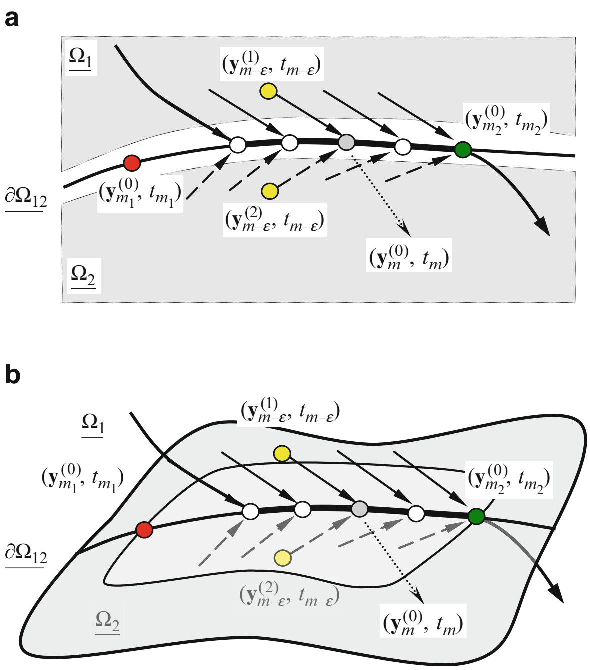

Fig. 3.8 (**a**) A cross-section view and (**b**) a three-dimensional view of the synchronization of resultant flows in vicinity of the constraint boundary $\partial\Omega_{12}$ in $(n_s + n_r)$-dimensional state space. On the constraint boundary, any point for synchronization is expressed by $(\mathbf{y}^{(0)}_m, t_m)$. In two domains, the resultant flows in the vicinity of the constraint boundary are expressed by $(\mathbf{y}^{(\alpha)}_{m-\varepsilon}, t_{m-\varepsilon})$ $(\alpha = 1, 2)$. The onset and vanishing points are $(\mathbf{y}^{(0)}_{m_1}, t_{m_1})$ and $(\mathbf{y}^{(0)}_{m_2}, t_{m_2})$ with *red* and *blue circular symbols*

So one obtains

$$(-1)^{\alpha}\varphi^{(2k_\alpha+2)}(\mathbf{y}^{(\alpha)}_{m_i\mp\varepsilon}, t_{m_i\mp\varepsilon}, \boldsymbol{\lambda}) < 0 \text{ and } (-1)^{\beta}\varphi^{(2k_\beta+2)}(\mathbf{y}^{(\beta)}_{m_i-}, t_{m_i-}, \boldsymbol{\lambda}) < 0.$$

i.e., Eq. (3.68) is obtained, vice versa. Therefore, this theorem is proved. □

In the foregoing theorem, the *onset and vanishing* conditions of the $(2k_\alpha : 2k_\beta)$-synchronization in Eqs. (3.67) and (3.68) for time $t = t_{m_i}$ $(i = 1, 2)$ are also the *vanishing and onset* conditions of the $(2k_\alpha : 2k_\beta)$-penetration and the $(2k_\alpha : 2k_\beta)$-desynchronization, respectively. To explain the synchronization of the two dynamical systems under the condition in Eq. (3.3) in the previous two theorems, such synchronization is sketched in Fig. 3.8. On the constraint boundary, any point for synchronization is expressed by $(\mathbf{y}^{(0)}_m, t_m)$. In the two domains, any flows in the vicinity of the boundary are expressed by $(\mathbf{y}^{(\alpha)}_{m-\varepsilon}, t_{m-\varepsilon})$ $(\alpha = 1, 2)$. The onset and vanishing points are $(\mathbf{y}^{(0)}_{m_1}, t_{m_1})$ and $(\mathbf{y}^{(0)}_{m_2}, t_{m_2})$ with red and blue circular symbols. Both of the two points belong to a submanifold on the boundary in the $(n_r + n_s)$-dimensional phase space. Once a flow of the resultant system of two dynamical systems from domain Ω_1 comes to any point of the subregion on the constraint boundary, the synchronization of the two dynamical systems to the constraint occurs until the point $(\mathbf{y}_{m_2}^{(0)}, t_{m_2})$is reached. If $t_{m_2} \to \infty$, such synchronization will not disappear forever. For $t_m > t_{m_1}$, once the resultant flows are on the constraint boundary, the synchronization of the two dynamical systems to the constraint will keep forever.

3.6 Desynchronization to Constraint

The synchronization for two dynamical systems to the constraint in Eq. (3.3) is discussed. The desynchronization of two dynamical systems is opposite to the synchronization. Similarly, for a case of $\mathbb{F}^{(\alpha)}(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\pi}^{(\alpha)}) = \mathbb{F}^{(0)}(\mathbf{y}^{(0)}, t, \boldsymbol{\lambda})$ on the constraint boundary, the desynchronization will be discussed, and the desynchronization for $\mathbb{F}^{(\alpha)}(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\pi}^{(\alpha)}) \neq \mathbb{F}^{(0)}(\mathbf{y}^{(0)}, t, \boldsymbol{\lambda})$ on the constraint boundary will be addressed. The desynchronization with $\mathbb{F}^{(\alpha)}(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\pi}^{(\alpha)}) = \mathbb{F}^{(0)}(\mathbf{y}^{(0)}, t, \boldsymbol{\lambda})$ is stated.

Theorem 3.5 *Consider two dynamical systems in Eqs. (3.1) and (3.2) with constraint in Eq. (3.3). For* $\mathbf{y}_{m\pm}^{(\alpha)} \in \Omega_\alpha$ $(\alpha \in \{1,2\})$ *and* $\mathbf{y}_m^{(0)} \in \partial\Omega_{12}$ *at time* t_m, $\mathbf{y}_{m\pm}^{(\alpha)} = \mathbf{y}_m^{(0)}$. *For any small* $\varepsilon > 0$, *there is a time interval* $[t_{m-\varepsilon}, t_{m+\varepsilon}]$. *At* $\mathbf{y}^{(\alpha)} \in \Omega_\alpha^{\pm\varepsilon}$ *for time* $t \in [t_{m-\varepsilon}, t_{m+\varepsilon}]$, *the constraint function* $\varphi(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\lambda})$ *is* C^{r_α}*-continuous* $(r_\alpha \geq 3)$ *and* $|\varphi^{(r_\alpha+1)}(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\lambda})| < \infty$. *For* $\mathbf{y}^{(\alpha)} \in \Omega_\alpha$ *and* $\mathbf{y}^{(0)} \in \partial\Omega_{12}$, *suppose* $D^{s_\alpha}\mathbb{F}^{(\alpha)}(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\pi}^{(\alpha)}) = D^{s_\alpha}\mathbb{F}^{(0)}(\mathbf{y}^{(0)}, t, \boldsymbol{\lambda})$ $(s_\alpha = 0, 1, 2, \cdots)$ *for* $\mathbf{y}^{(\alpha)} = \mathbf{y}^{(0)}$. *The two dynamical systems in Eqs. (3.1) and (3.2) to the constraint in Eq. (3.3) are desynchronized for time* $t \in [t_{m_1}, t_{m_2}]$ *if and only if*

(i) *for* $\mathbf{y}_m^{(\alpha)} \in \Omega_\alpha$ *and* $\mathbf{y}_m^{(0)} \in \partial\Omega_{12}$ *with any time* t_m

$$\begin{aligned} &\mathbf{y}_m^{(\alpha)} = \mathbf{y}_m^{(0)}, \varphi^{(r_\alpha)}(\mathbf{y}_m^{(\alpha)}, t_m, \boldsymbol{\lambda}) = 0 \\ &\text{for } \alpha = 1,2 \text{ and } r_\alpha = 0,1,2,\cdots \end{aligned} \tag{3.69}$$

(ii) *for* $\mathbf{y}_\kappa^{(\alpha)} \in \Omega_\alpha^{+\varepsilon}$ *at time* $t_\kappa^+ \in (t_m, t_{m+\varepsilon}]$ *and* $\mathbf{y}_m^{(0)} \in \partial\Omega_{12}$ *with* $t_m \in (t_{m_1}, t_{m_2})$

$$\begin{aligned} &\mathbf{y}_\kappa^{(\alpha)} \neq \mathbf{y}_m^{(0)}, (-1)^\alpha \varphi^{(1)}(\mathbf{y}_\kappa^{(\alpha)}, t_\kappa^+, \boldsymbol{\lambda}) < 0, \\ &\lim_{t_\kappa^+ \to t_m} \varphi^{(1)}(\mathbf{y}_\kappa^{(\alpha)}, t_\kappa^+, \boldsymbol{\lambda}) = 0 \text{ for } \alpha = 1,2 \end{aligned} \tag{3.70}$$

(iii) *for* $\mathbf{y}_\kappa^{(\alpha)} \in \Omega_\alpha^{-\varepsilon}$ *at time* $t_\kappa^- \in [t_{m-\varepsilon}, t_m)$ *and* $\mathbf{y}_m^{(0)} \in \partial\Omega_{12}$ *with* $t_m \notin [t_{m_1}, t_{m_2}]$

$$\begin{aligned} &\mathbf{y}_\kappa^{(\alpha)} \neq \mathbf{y}_m^{(0)}, (-1)^\alpha \varphi^{(1)}(\mathbf{y}_\kappa^{(\alpha)}, t_\kappa^-, \boldsymbol{\lambda}) > 0, \\ &\lim_{t_\kappa^- \to t_m} \varphi^{(1)}(\mathbf{y}_\kappa^{(\alpha)}, t_\kappa^-, \boldsymbol{\lambda}) = 0 \text{ for } \alpha = 1,2 \end{aligned} \tag{3.71}$$

(iv) *for* $\mathbf{y}_{\kappa}^{(\alpha)} \in \boldsymbol{\Omega}_{\alpha}^{+\varepsilon}$ *at time* $t_{\kappa}^{-} \in [t_{m-\varepsilon}, t_{m-})$ *or* $t_{\kappa}^{+} \in (t_{m+}, t_{m+\varepsilon}]$ *and* $\mathbf{y}_{m}^{(0)} \in \partial\boldsymbol{\Omega}_{12}$ *with* $t_m = t_{m_1}$ and t_{m_2}

$$\begin{aligned} &\mathbf{y}_{\kappa}^{(\alpha)} \neq \mathbf{y}_{m}^{(0)}, \lim_{t_{\kappa}^{\pm} \to t_{m\pm}} \varphi^{(1)}(\mathbf{y}_{\kappa}^{(\alpha)}, t_{\kappa}^{\pm}, \boldsymbol{\lambda}) > 0, \\ &\lim_{t_{\kappa}^{\pm} \to t_{m\pm}} (-1)^{\alpha} \varphi^{(2)}(\mathbf{y}_{\kappa}^{(\alpha)}, t_{\kappa}^{\pm}, \boldsymbol{\lambda}) < 0 \text{ for } \alpha = 1, 2 \end{aligned} \tag{3.72}$$

Proof Once Definitions 3.13, 3.14, 3.17, and 3.18 are used, the proof of this theorem is similar to the proof of Theorem 3.1. □

Theorem 3.6 *Consider two dynamical systems in Eqs. (3.1) and (3.2) with constraint in Eq. (3.3). For* $\mathbf{y}_{m\pm}^{(\alpha)} \in \boldsymbol{\Omega}_{\alpha}$ $(\alpha \in \{1,2\})$ *and* $\mathbf{y}_{m}^{(0)} \in \partial\boldsymbol{\Omega}_{12}$ *at time* t_m, $\mathbf{y}_{m\pm}^{(\alpha)} = \mathbf{y}_{m}^{(0)}$. *For any small* $\varepsilon > 0$, *there is a time interval* $[t_{m-\varepsilon}, t_{m+\varepsilon}]$. *At* $\mathbf{y}^{(\alpha)} \in \boldsymbol{\Omega}_{\alpha}^{\pm\varepsilon}$ *for time* $t \in [t_{m-\varepsilon}, t_{m+\varepsilon}]$, *the constraint function* $\varphi(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\lambda})$ *is* C^{r_α}*-continuous* $(r_\alpha \geq 2k_\alpha + 1)$ *and* $|\varphi^{(r_\alpha+1)}(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\lambda})| < \infty$. *For* $\mathbf{y}^{(\alpha)} \in \boldsymbol{\Omega}_{\alpha}$ *and* $\mathbf{y}^{(0)} \in \partial\boldsymbol{\Omega}_{12}$, *suppose* $D^{s_\alpha}\mathbb{F}^{(\alpha)}(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\pi}^{(\alpha)}) = D^{s_\alpha}\mathbb{F}^{(0)}(\mathbf{y}^{(0)}, t, \boldsymbol{\lambda})$ $(s_\alpha = 0, 1, 2, \cdots)$ *for* $\mathbf{y}^{(\alpha)} = \mathbf{y}^{(0)}$. *The two dynamical systems in Eqs. (3.1) and (3.2) to constraint in Eq. (3.3) are desynchronized for time* $t \in [t_{m_1}, t_{m_2}]$ *if and only if*

(i) *for* $\mathbf{y}_{m\pm}^{(\alpha)} \in \boldsymbol{\Omega}_{\alpha}$ *and* $\mathbf{y}_{m}^{(0)} \in \partial\boldsymbol{\Omega}_{12}$ *with any time* t_m

$$\begin{aligned} &\mathbf{y}_{m\pm}^{(\alpha)} = \mathbf{y}_{m}^{(0)}, \varphi^{(r_\alpha)}(\mathbf{y}_{m\pm}^{(\alpha)}, t_m, \boldsymbol{\lambda}) = 0 \\ &\text{for } \alpha = 1, 2 \text{ and } r_\alpha = 0, 1, 2, \cdots \end{aligned} \tag{3.73}$$

(ii) *for* $\mathbf{y}_{\kappa}^{(\alpha)} \in \boldsymbol{\Omega}_{\alpha}^{+\varepsilon}$ *at time* $t_{\kappa}^{+} \in (t_m, t_{m+\varepsilon}]$ *and* $\mathbf{y}_{m}^{(0)} \in \partial\boldsymbol{\Omega}_{12}$ *with* $t_m \in (t_{m_1}, t_{m_2})$

$$\begin{aligned} &\mathbf{y}_{\kappa}^{(\alpha)} \neq \mathbf{y}_{m}^{(0)}, \lim_{t_{\kappa}^{+} \to t_{m+}} \varphi^{(s_\alpha)}(\mathbf{y}_{\kappa}^{(\alpha)}, t_{\kappa}^{+}, \boldsymbol{\lambda}) = 0 \text{ for } s_\alpha = 1, 2, \cdots, 2k_\alpha; \\ &(-1)^{\alpha} \varphi^{(2k_\alpha+1)}(\mathbf{y}_{\kappa}^{(\alpha)}, t_{\kappa}^{+}, \boldsymbol{\lambda}) < 0 \text{ and} \\ &\lim_{t_{\kappa}^{+} \to t_m} \varphi^{(2k_\alpha+1)}(\mathbf{y}_{\kappa}^{(\alpha)}, t_{\kappa}^{+}, \boldsymbol{\lambda}) = 0 \text{ for } \alpha = 1, 2 \end{aligned} \tag{3.74}$$

(iii) *for* $\mathbf{y}_{\kappa}^{(\alpha)} \in \boldsymbol{\Omega}_{\alpha}^{-\varepsilon}$ *at time* $t_{\kappa}^{-} \in [t_{m-\varepsilon}, t_m)$ *and* $\mathbf{y}_{m}^{(0)} \in \partial\boldsymbol{\Omega}_{12}$ *with* $t_m \notin [t_{m_1}, t_{m_2}]$

$$\begin{aligned} &\mathbf{y}_{\kappa}^{(\alpha)} \neq \mathbf{y}_{m}^{(0)}, \lim_{t_{\kappa}^{-} \to t_{m-}} \varphi^{(s_\alpha)}(\mathbf{y}_{\kappa}^{(\alpha)}, t_{\kappa}^{-}, \boldsymbol{\lambda}) = 0 \text{ for } s_\alpha = 1, 2, \cdots, 2k_\alpha; \\ &(-1)^{\alpha} \varphi^{(2k_\alpha+1)}(\mathbf{y}_{\kappa}^{(\alpha)}, t_{\kappa}^{-}, \boldsymbol{\lambda}) > 0 \text{ and} \\ &\lim_{t_{\kappa}^{-} \to t_m} \varphi^{(2k_\alpha+1)}(\mathbf{y}_{\kappa}^{(\alpha)}, t_{\kappa}^{-}, \boldsymbol{\lambda}) = 0 \text{ for } \alpha = 1, 2 \end{aligned} \tag{3.75}$$

(iv) *for* $\mathbf{y}_{\kappa}^{(\alpha)} \in \boldsymbol{\Omega}_{\alpha}^{+\varepsilon}$ *at time* $t_{\kappa}^{-} \in [t_{m-\varepsilon}, t_{m-})$ *or* $t_{\kappa}^{+} \in (t_{m+}, t_{m+\varepsilon}]$ *and* $\mathbf{y}_{m}^{(0)} \in \partial\boldsymbol{\Omega}_{12}$ *with* $t_m = t_{m_1}$ and t_{m_2}

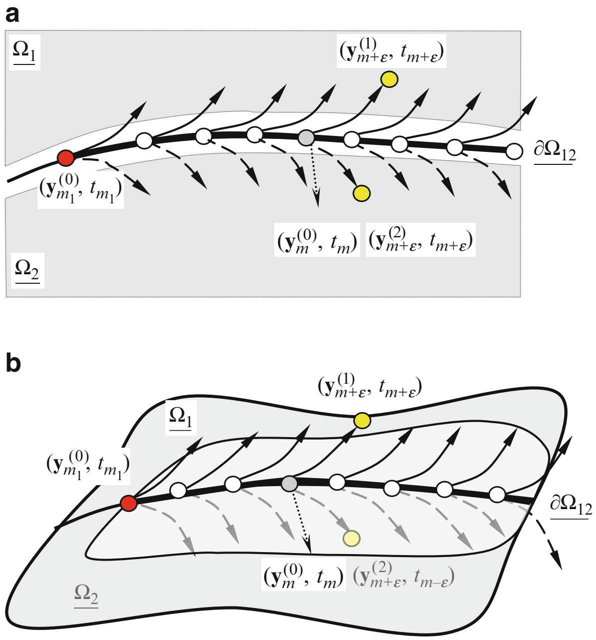

Fig. 3.9 (**a**) Cross-section view and (**b**) three-dimensional view for the desynchronization of slave and master flows in vicinity of the boundary $\partial\Omega_{12}$ in $(n_r + n_s)$-dimensional state space. On the boundary, any point for desynchronization is expressed by $(\mathbf{y}_m^{(0)}, t_m)$. In the two domains, the flows in the vicinity of the boundary are expressed by $(\mathbf{y}_{m+\varepsilon}^{(\alpha)}, t_{m+\varepsilon})$ $(\alpha = 1, 2)$. The onset point is $(\mathbf{y}_{m_1}^{(0)}, t_{m_1})$, depicted by a *red circular symbol*

$$
\begin{aligned}
&\mathbf{y}_\kappa^{(\alpha)} \neq \mathbf{y}_m^{(0)}, \lim_{t_\kappa^\pm \to t_{m\pm}} \varphi^{(s_\alpha)}(\mathbf{y}_\kappa^{(\alpha)}, t_\kappa^\pm, \boldsymbol{\lambda}) = 0 \text{ for } s_\alpha = 1, 2, \cdots, 2k_\alpha + 1; \\
&\lim_{t_\kappa^\pm \to t_{m\pm}} (-1)^\alpha \varphi^{(2k_\alpha+2)}(\mathbf{y}_\kappa^{(\alpha)}, t_\kappa^\pm, \boldsymbol{\lambda}) < 0 \text{ for } \alpha = 1, 2
\end{aligned} \tag{3.76}
$$

Proof Once Definitions 3.21, 3.22, 3.25, and 3.26 are used, the proof of this theorem is similar to the proof of Theorem 3.2. □

If $t_{m_1} \to -\infty$ and $t_{m_2} \to \infty$, such a desynchronization of two dynamical systems to constraint in Eq. (3.3) is absolute. Once the resultant flows on the constraint boundary are repelled, such a desynchronization can keep forever. To explain the two foregoing theorems, the desynchronization of two dynamical systems to a specific constraint is sketched in Fig. 3.9 through the resultant flows in the vicinity of the constraint boundary $\partial\Omega_{12}$. Any point for desynchronization on the constraint boundary is expressed by $(\mathbf{y}_m^{(0)}, t_m)$. In the two domains, the resultant flows in the vicinity of the boundary are expressed by $(\mathbf{y}_{m+\varepsilon}^{(\alpha)}, t_{m+\varepsilon})$ $(\alpha = 1, 2)$. The onset point for the desynchronization is denoted by $(\mathbf{y}_{m_1}^{(0)}, t_{m_1})$. For $t_m > t_{m_1}$ and $t_m \to \infty$, all the resultant flows leave from the constraint boundary. However, if $t_{m_2} > t_{m_1}$ is finite, such desynchronization to the constraint will disappear at a point $(\mathbf{y}_{m_2}^{(0)}, t_{m_2})$.

For $\mathbb{F}^{(\alpha)}(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\pi}^{(\alpha)}) = \mathbb{F}^{(0)}(\mathbf{y}^{(0)}, t, \boldsymbol{\lambda})$, the desynchronization of two dynamical systems to a specific constraint is different from those for $\mathbb{F}^{(\alpha)}(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\pi}^{(\alpha)}) \neq \mathbb{F}^{(0)}(\mathbf{y}^{(0)}, t, \boldsymbol{\lambda})$. Thus, the conditions for the desynchronization of two dynamical systems with discontinuous vector fields are discussed as follows.

Theorem 3.7 *Consider two dynamical systems in Eqs. (3.1) and (3.2) with constraint in Eq. (3.3). For* $\mathbf{y}_{m\pm}^{(\alpha)} \in \Omega_\alpha$ *(*$\alpha \in \{1,2\}$*) and* $\mathbf{y}_m^{(0)} \in \partial\Omega_{12}$ *at time* t_m*,* $\mathbf{y}_{m\pm}^{(\alpha)} = \mathbf{y}_m^{(0)}$*. For any small* $\varepsilon > 0$*, there is a time interval* $[t_{m-\varepsilon}, t_{m+\varepsilon}]$*. At* $\mathbf{y}^{(\alpha)} \in \Omega_\alpha^{\pm\varepsilon}$ *for time* $t \in [t_{m-\varepsilon}, t_{m+\varepsilon}]$*, the constraint function* $\varphi(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\lambda})$ *is* C^{r_α}*-continuous (*$r_\alpha \geq 3$*) and* $|\varphi^{(r_\alpha+1)}(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\lambda})| < \infty$*. For* $\mathbf{y}^{(\alpha)} \in \Omega_\alpha$ *and* $\mathbf{y}^{(0)} \in \partial\Omega_{12}$*, suppose* $\mathbb{F}^{(\alpha)}(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\pi}^{(\alpha)}) \neq \mathbb{F}^{(0)}(\mathbf{y}^{(0)}, t, \boldsymbol{\lambda})$ *for* $\mathbf{y}^{(\alpha)} = \mathbf{y}^{(0)}$*. The two dynamical systems in Eqs. (3.1) and (3.2) to the constraint in Eq. (3.3) are desynchronized for time* $t \in [t_{m_1}, t_{m_2}]$ *if and only if*

(i) *for* $\mathbf{y}_{m+}^{(\alpha)} = \mathbf{y}_m^{(0)}$ *and* $\mathbf{y}^{(\alpha)} \in \Omega_\alpha$ *(*$\alpha \in \{1,2\}$*) at time* $t = t_m \in [t_{m_1}, t_{m_2}]$

$$\varphi(\mathbf{y}_{m+}^{(\alpha)}, t_{m+}, \boldsymbol{\lambda}) = \varphi(\mathbf{y}_m^{(0)}, t_m, \boldsymbol{\lambda}) = 0 \tag{3.77}$$

(ii) *for time* $t_m \in [t_{m_1}, t_{m_2})$*,*

$$\mathbf{y}_{m+}^{(\alpha)} = \mathbf{y}_m^{(0)} \text{ and } (-1)^\alpha \varphi^{(1)}(\mathbf{y}_{m+}^{(\alpha)}, t_{m+}, \boldsymbol{\lambda}) < 0 \text{ for } \alpha = 1, 2 \tag{3.78}$$

(iii) *with an penetration for time* $t = t_{m_i}$*,* $\mathbf{y}_{m_i\pm}^{(\alpha)} = \mathbf{y}_{m_i}^{(0)} = \mathbf{y}_{m_i+}^{(\beta)}$ *(*$i = 1, 2$*),*

$$\begin{aligned} &\varphi^{(1)}(\mathbf{y}_{m_i\pm}^{(\alpha)}, t_{m_i\pm}, \boldsymbol{\lambda}) = 0 \text{ and } (-1)^\alpha \varphi^{(2)}(\mathbf{y}_{m_i\pm}^{(\alpha)}, t_{m_i\pm}, \boldsymbol{\lambda}) < 0, \\ &(-1)^\beta \varphi^{(1)}(\mathbf{y}_{m_i+}^{(\beta)}, t_{m_i+}, \boldsymbol{\lambda}) < 0 \text{ for } \alpha, \beta \in \{1,2\} \text{ and } \beta \neq \alpha, \end{aligned} \tag{3.79}$$

or with a synchronization for time $t = t_{m_i}$*,* $\mathbf{y}_{m_i\pm}^{(\alpha)} = \mathbf{y}_{m_i}^{(0)} = \mathbf{y}_{m_i\pm}^{(\beta)}$*,*

$$\begin{aligned} &\varphi^{(1)}(\mathbf{y}_{m_i\pm}^{(\alpha)}, t_{m_i\pm}, \boldsymbol{\lambda}) = 0 \text{ and } (-1)^\alpha \varphi^{(2)}(\mathbf{y}_{m_i\pm}^{(\alpha)}, t_{m_i\pm}, \boldsymbol{\lambda}) < 0, \\ &\varphi^{(1)}(\mathbf{y}_{m_i\pm}^{(\beta)}, t_{m_i\pm}, \boldsymbol{\lambda}) = 0 \text{ and } (-1)^\beta \varphi^{(2)}(\mathbf{y}_{m_i\pm}^{(\beta)}, t_{m_i\pm}, \boldsymbol{\lambda}) < 0 \\ &\text{for } \alpha, \beta \in \{1,2\} \text{ and } \beta \neq \alpha. \end{aligned} \tag{3.80}$$

Proof By using Definitions 3.13, 3.17–3.19, the proof of this theorem is similar to the proof of Theorem 3.3. □

From the foregoing theorem, the desynchronization of two dynamical systems to a specific constraint requires that the first-order derivative of the constraint function be greater than zero. In addition, the *onset and vanishing* conditions of desynchronization in Eqs. (3.79) and (3.80) are the *vanishing and onset* conditions for onset of the penetration and synchronization with the desynchronization, respectively. The following theorem will give the corresponding conditions for the desynchronization of two dynamical systems to a specific constraint with the higher order singularity.

Theorem 3.8 *Consider two dynamical systems in Eqs. (3.1) and (3.2) with constraint in Eq. (3.3). For* $\mathbf{y}_{m\pm}^{(\alpha)} \in \Omega_\alpha$ *(*$\alpha \in \{1,2\}$*) and* $\mathbf{y}_m^{(0)} \in \partial\Omega_{12}$ *at time* t_m*,* $\mathbf{y}_{m\pm}^{(\alpha)} = \mathbf{y}_m^{(0)}$*. For any small* $\varepsilon > 0$*, there is a time interval* $[t_{m-\varepsilon}, t_{m+\varepsilon}]$*. At* $\mathbf{y}^{(\alpha)} \in \Omega_\alpha^{\pm\varepsilon}$ *for time* $t \in [t_{m-\varepsilon}, t_{m+\varepsilon}]$*, the constraint function* $\varphi(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\lambda})$ *is* C^{r_α}*-continuous (*$r_\alpha \geq 2k_\alpha + 1$*) and* $|\varphi^{(r_\alpha+1)}(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\lambda})| < \infty$*. For* $\mathbf{y}^{(\alpha)} \in \Omega_\alpha$ *and* $\mathbf{y}^{(0)} \in \partial\Omega_{12}$*, suppose* $\mathbb{F}^{(\alpha)}(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\pi}^{(\alpha)}) \neq \mathbb{F}^{(0)}(\mathbf{y}^{(0)}, t, \boldsymbol{\lambda})$ *for* $\mathbf{y}^{(\alpha)} = \mathbf{y}^{(0)}$*. The two dynamical systems in Eqs. (3.1) and (3.2) to constraint in Eq. (3.3) are desynchronized of the* $(2k_1 : 2k_2)$*-type for time* $t \in [t_{m_1}, t_{m_2}]$ *if and only if*

(i) *for* $\mathbf{y}_{m+}^{(\alpha)} = \mathbf{y}_m^{(0)}$ *and* $\mathbf{y}^{(\alpha)} \in \Omega_\alpha$ *(*$\alpha \in \{1,2\}$*) at time* $t = t_m \in [t_{m_1}, t_{m_2}]$

$$\varphi(\mathbf{y}_{m+}^{(\alpha)}, t_{m+}, \boldsymbol{\lambda}) = \varphi(\mathbf{y}_m^{(0)}, t_m, \boldsymbol{\lambda}) = 0 \tag{3.81}$$

(ii) *for time* $t_m \in (t_{m_1}, t_{m_2})$*,* $\mathbf{y}_{m+}^{(\alpha)} = \mathbf{y}_m^{(0)} = \mathbf{y}_{m+}^{(\beta)}$

$$\begin{aligned}
&\varphi^{(s_\alpha)}(\mathbf{y}_{m+}^{(\alpha)}, t_{m+}, \boldsymbol{\lambda}) < 0 \quad (s_\alpha = 1, 2, \cdots, 2k_\alpha),\\
&(-1)^\alpha \varphi^{(2k_\alpha+1)}(\mathbf{y}_{m+}^{(\alpha)}, t_{m+}, \boldsymbol{\lambda}) < 0,\\
&\varphi^{(s_\beta)}(\mathbf{y}_{m+}^{(\beta)}, t_{m+}, \boldsymbol{\lambda}) = 0 \quad (s_\beta = 1, 2, \cdots, 2k_\beta),\\
&(-1)^\beta \varphi^{(2k_\beta+1)}(\mathbf{y}_{m+}^{(\beta)}, t_{m+}, \boldsymbol{\lambda}) < 0 \quad \text{for } \beta \in \{1,2\} \text{ and } \alpha \neq \beta
\end{aligned} \tag{3.82}$$

(iii) *with a* $(2k_\alpha : 2k_\beta)$*-penetration flow for time* $t = t_{m_i}$*,* $\mathbf{y}_{m_i\pm}^{(\alpha)} = \mathbf{y}_{m_i}^{(0)} = \mathbf{y}_{m_i+}^{(\beta)}$

$$\begin{aligned}
&\varphi^{(s_\alpha)}(\mathbf{y}_{m_i\pm}^{(\alpha)}, t_{m_i\pm}, \boldsymbol{\lambda}) = 0 \ (s_\alpha = 1, 2, \cdots, 2k_\alpha + 1),\\
&(-1)^\alpha \varphi^{(2k_\alpha+2)}(\mathbf{y}_{m+}^{(\alpha)}, t_{m+}, \boldsymbol{\lambda}) < 0,\\
&\varphi^{(s_\beta)}(\mathbf{y}_{m_i+}^{(\beta)}, t_{m_i+}, \boldsymbol{\lambda}) = 0 \ (s_\beta = 1, 2, \cdots, 2k_\beta),\\
&(-1)^\beta \varphi^{(2k_\beta+1)}(\mathbf{y}_{m_i+}^{(\beta)}, t_{m_i+}, \boldsymbol{\lambda}) < 0 \quad \text{for } \alpha,\beta \in \{1,2\} \text{ and } \alpha \neq \beta
\end{aligned} \tag{3.83}$$

or with a $(2k_1 : 2k_2)$*-synchronization for time* $t = t_{m_i}$*,* $\mathbf{y}_{m_i\pm}^{(\alpha)} = \mathbf{y}_{m_i}^{(0)} = \mathbf{y}_{m_i\pm}^{(\beta)}$

$$\begin{aligned}
&\varphi^{(s_\alpha)}(\mathbf{y}_{m_i\pm}^{(\alpha)}, t_{m_i\pm}, \boldsymbol{\lambda}) = 0 \quad (s_\alpha = 1, 2, \cdots, 2k_\alpha + 1),\\
&(-1)^\alpha \varphi^{(2k_\alpha+2)}(\mathbf{y}_{m+}^{(\alpha)}, t_{m+}, \boldsymbol{\lambda}) < 0,\\
&\varphi^{(s_\beta)}(\mathbf{y}_{m_i\pm}^{(\beta)}, t_{m_i\pm}, \boldsymbol{\lambda}) = 0 \quad (s_\beta = 1, 2, \cdots, 2k_\beta + 1),\\
&(-1)^\beta \varphi^{(2k_\beta+2)}(\mathbf{y}_{m_i\pm}^{(\beta)}, t_{m_i\pm}, \boldsymbol{\lambda}) < 0 \quad \text{for } \alpha,\beta \in \{1,2\} \text{ and } \alpha \neq \beta.
\end{aligned} \tag{3.84}$$

Proof Using Definitions 3.23, 3.25–3.27, the proof of this theorem is similar to Theorem 3.4. □

The *onset and vanishing* conditions of the $(2k_1 : 2k_2)$-desynchronization in Eqs. (3.83) and (3.84) are the *vanishing and onset* conditions of the $(2k_\alpha : 2k_\beta)$

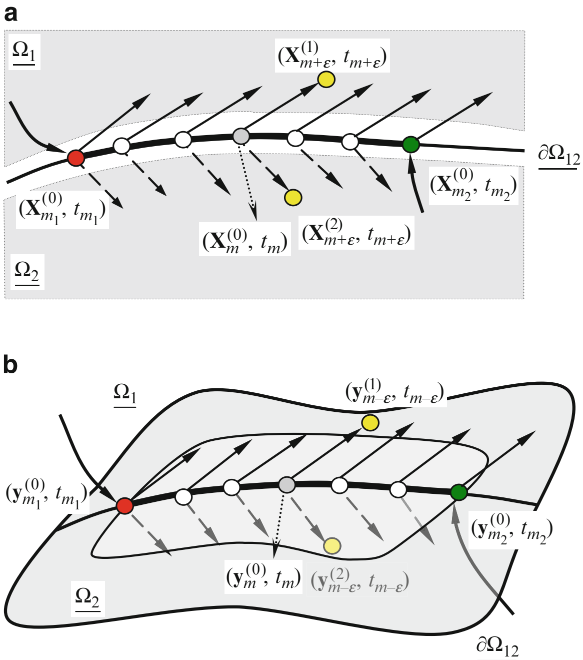

Fig. 3.10 (**a**) A cross-section view and (**b**) a three-dimensional view of the desynchronization of resultant flows in vicinity of the constraint boundary $\partial\Omega_{12}$ in $(n_r + n_s)$-dimensional state space. On the constant boundary, any point for desynchronization is expressed by $(\mathbf{y}^{(0)}_m, t_m)$. In two domains, the resultant flows in the vicinity of the constant boundary are expressed by $(\mathbf{y}^{(\alpha)}_{m-\varepsilon}, t_{m-\varepsilon})$ $(\alpha = 1, 2)$. The onset and vanishing points are $(\mathbf{y}^{(0)}_{m_1}, t_{m_1})$ and $(\mathbf{y}^{(0)}_{m_2}, t_{m_2})$ with *red* and *green circular symbols*

penetration and the $(2k_1 : 2k_2)$-synchronization, respectively. The $(2k_1 : 2k_2)$-desynchronization requires that all the $(2k_1 + 1 : 2k_2 + 1)$-order derivative of the constraint function should be greater than zero. The desynchronization of two dynamical systems to a specific constraint is presented in the previous two theorems, as sketched in Fig. 3.10 through the resultant flows in the vicinity of the constraint boundary. On the constraint boundary, any point relative to desynchronization is expressed by $(\mathbf{y}^{(0)}_m, t_m)$. In the two domains, the flows in the vicinity of the constraint boundary are expressed by $(\mathbf{y}^{(\alpha)}_{m+\varepsilon}, t_{m+\varepsilon})$ $(\alpha = 1, 2)$. The *onset and vanishing* points are $(\mathbf{y}^{(0)}_{m_1}, t_{m_1})$ and $(\mathbf{y}^{(0)}_{m_2}, t_{m_2})$ with red and green circular symbols, which are generated by the two penetrations. The points $(\mathbf{y}^{(0)}_{m_1}, t_{m_1})$ and $(\mathbf{y}^{(0)}_{m_2}, t_{m_2})$are starting and vanishing points of the resultant flow relative to desynchronization.

If $t_{m_2} \to \infty$, once the desynchronization exists, no any synchronization of two systems to a specific constraint can be achieved. For a case of $\mathbb{F}^{(\alpha)}(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\pi}^{(\alpha)}) \neq \mathbb{F}^{(0)}(\mathbf{y}^{(0)}, t, \boldsymbol{\lambda})$ and $\mathbb{F}^{(\alpha)}(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\pi}^{(\alpha)}) \neq \mathbb{F}^{(\beta)}(\mathbf{y}^{(\beta)}, t, \boldsymbol{\pi}^{(\beta)})$, the desynchronization can be determined through the two foregoing theorems.

3.7 Penetration to Constraint

The synchronization and desynchronization of two dynamical systems to a specific constraint have been discussed. The penetration of two dynamical systems to a specific constraint is also very important for the onset and vanishing of synchronization and desynchronization. For $\mathbb{F}^{(\alpha)}(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\pi}^{(\alpha)}) = \mathbb{F}^{(0)}(\mathbf{y}^{(0)}, t, \boldsymbol{\lambda})$ with $\alpha = 1, 2$, the penetration of two dynamical systems to a specific constraint cannot exist. However, if two dynamical systems to a specific constraint possess discontinuous vector fields, the penetration can occur at the constraint boundary. The corresponding theorems are presented as follows.

Theorem 3.9 *Consider two dynamical systems in Eqs. (3.1) and (3.2) with constraint in Eq. (3.3). For* $\mathbf{y}_{m\pm}^{(\alpha)} \in \Omega_\alpha$ *(*$\alpha \in \{1,2\}$*) and* $\mathbf{y}_m^{(0)} \in \partial\Omega_{12}$ *at time* t_m*,* $\mathbf{y}_{m\pm}^{(\alpha)} = \mathbf{y}_m^{(0)}$*. For any small* $\varepsilon > 0$*, there is a time interval* $[t_{m-\varepsilon}, t_{m+\varepsilon}]$*. At* $\mathbf{y}^{(\alpha)} \in \Omega_\alpha^{\pm\varepsilon}$ *for time* $t \in [t_{m-\varepsilon}, t_{m+\varepsilon}]$*, the constraint function* $\varphi(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\lambda})$ *is* C^{r_α}*-continuous (*$r_\alpha \geq 3$*) and* $|\varphi^{(r_\alpha+1)}(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\lambda})| < \infty$*. For* $\mathbf{y}^{(\alpha)} \in \Omega_\alpha$ *and* $\mathbf{y}^{(0)} \in \partial\Omega_{12}$*, suppose* $D^{s_\alpha}\mathbb{F}^{(\alpha)}(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\pi}^{(\alpha)}) \neq D^{s_\alpha}\mathbb{F}^{(0)}(\mathbf{y}^{(0)}, t, \boldsymbol{\lambda})$ s_α *(*$= 0, 1, 2, \cdots$*) for* $\mathbf{y}^{(\alpha)} = \mathbf{y}^{(0)}$*. The two dynamical systems in Eqs. (3.1) and (3.2) to the constraint in Eq. (3.3) is penetrated at time* $t \in [t_{m_1}, t_{m_2}]$ *if and only if*

(i) *for* $\mathbf{y}_{m\pm}^{(\alpha)} = \mathbf{y}_m^{(0)}$ *and* $\mathbf{y}^{(\alpha)} \in \Omega_\alpha$ *(*$\alpha \in \{1,2\}$*) at time* $t = t_m \in [t_{m_1}, t_{m_2}]$

$$\varphi(\mathbf{y}_{m\pm}^{(\alpha)}, t_{m\pm}, \boldsymbol{\lambda}) = \varphi(\mathbf{y}_m^{(0)}, t_m, \boldsymbol{\lambda}) = 0 \tag{3.85}$$

(ii) *at time* $t = t_m \in (t_{m_1}, t_{m_2})$*,* $\mathbf{y}_{m-}^{(\alpha)} = \mathbf{y}_m^{(0)} = \mathbf{y}_{m+}^{(\beta)}$

$$\begin{aligned} &(-1)^\alpha \varphi^{(1)}(\mathbf{y}_{m-}^{(\alpha)}, t_{m-}, \boldsymbol{\lambda}) > 0 \text{ and } (-1)^\beta \varphi^{(1)}(\mathbf{y}_{m+}^{(\beta)}, t_{m+}, \boldsymbol{\lambda}) < 0 \\ &\text{for } \alpha, \beta \in \{1,2\} \text{ and } \alpha \neq \beta. \end{aligned} \tag{3.86}$$

(iii) *with a synchronization at time* $t = t_{m_i}$*,* $\mathbf{y}_{m_i-}^{(\alpha)} = \mathbf{y}_{m_i}^{(0)} = \mathbf{y}_{m_i\pm}^{(\beta)}$ *(*$i \in \{1,2\}$*),*

$$\begin{aligned} &(-1)^\alpha \varphi^{(1)}(\mathbf{y}_{m_i-}^{(\alpha)}, t_{m_i-}, \boldsymbol{\lambda}) > 0, \\ &\varphi^{(1)}(\mathbf{y}_{m_i\pm}^{(\beta)}, t_{m_i\pm}, \boldsymbol{\lambda}) = 0 \text{ and } (-1)^\beta \varphi^{(2)}(\mathbf{y}_{m_i\pm}^{(\beta)}, t_{m_i\pm}, \boldsymbol{\lambda}) < 0 \\ &\text{for } \alpha, \beta \in \{1,2\} \text{ and } \alpha \neq \beta, \end{aligned} \tag{3.87}$$

or with a desynchronization at time $t = t_{m_i}$, $\mathbf{y}_{m_i\mp}^{(\alpha)} = \mathbf{y}_{m_i}^{(0)} = \mathbf{y}_{m_i+}^{(\beta)}$ ($i \in \{1,2\}$),

$$\begin{aligned} &\varphi^{(1)}(\mathbf{y}_{m_i\mp}^{(\alpha)}, t_{m_i\mp}, \boldsymbol{\lambda}) = 0, \text{ and } (-1)^\alpha \varphi^{(2)}(\mathbf{y}_{m_i\mp}^{(\alpha)}, t_{m_i\mp}, \boldsymbol{\lambda}) < 0 \\ &(-1)^\beta \varphi^{(1)}(\mathbf{y}_{m_i+}^{(\beta)}, t_{m_i+}, \boldsymbol{\lambda}) < 0 \text{ for } \alpha, \beta \in \{1,2\} \text{ and } \alpha \neq \beta, \end{aligned} \tag{3.88}$$

or with a switching penetration at time $t = t_{m_i}, \mathbf{y}_{m_i\mp}^{(\alpha)} = \mathbf{y}_{m_i}^{(0)} = \mathbf{y}_{m_i\pm}^{(\beta)}$ ($i \in \{1,2\}$)

$$\varphi^{(1)}(\mathbf{y}^{(\alpha)}_{m_i\mp}, t_{m_i\mp}, \boldsymbol{\lambda}) = 0, \text{ and } (-1)^{\alpha}\varphi^{(2)}(\mathbf{y}^{(\alpha)}_{m_i\mp}, t_{m_i\mp}, \boldsymbol{\lambda}) < 0,$$
$$\varphi^{(1)}(\mathbf{y}^{(\beta)}_{m_i\pm}, t_{m_i\pm}, \boldsymbol{\lambda}) = 0 \text{ and } (-1)^{\beta}\varphi^{(2)}(\mathbf{y}^{(\beta)}_{m_i\pm}, t_{m_i\pm}, \boldsymbol{\lambda}) < 0$$
$$\text{for } \alpha, \beta \in \{1,2\} \text{ and } \alpha \neq \beta. \tag{3.89}$$

Proof By using Definitions 3.15, 3.17, 3.18, and 3.20, the proof of this theorem is similar to the proof of Theorem 3.3. □

Theorem 3.10 *Consider two dynamical systems in Eqs. (3.1) and (3.2) with constraint in Eq. (3.3). For* $\mathbf{y}^{(\alpha)}_{m\pm} \in \boldsymbol{\Omega}_{\alpha}$ *(*$\alpha \in \{1,2\}$*) and* $\mathbf{y}^{(0)}_{m} \in \partial\boldsymbol{\Omega}_{12}$ *at time* t_m*,* $\mathbf{y}^{(\alpha)}_{m\pm} = \mathbf{y}^{(0)}_{m}$*. For any small* $\varepsilon > 0$*, there is a time interval* $[t_{m-\varepsilon}, t_{m+\varepsilon}]$*. At* $\mathbf{y}^{(\alpha)} \in \boldsymbol{\Omega}^{\pm\varepsilon}_{\alpha}$ *for time* $t \in [t_{m-\varepsilon}, t_{m+\varepsilon}]$*, the constraint function* $\varphi(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\lambda})$ *is* C^{r_α}*-continuous (*$r_\alpha \geq 2k_\alpha + 1$*) and* $|\varphi^{(r_\alpha+1)}(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\lambda})| < \infty$*. For* $\mathbf{y}^{(\alpha)} \in \boldsymbol{\Omega}_{\alpha}$ *and* $\mathbf{y}^{(0)} \in \partial\boldsymbol{\Omega}_{12}$*, suppose* $\mathbb{F}^{(\alpha)}(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\pi}^{(\alpha)}) \neq \mathbb{F}^{(0)}(\mathbf{y}^{(0)}, t, \boldsymbol{\lambda})$ *for* $\mathbf{y}^{(\alpha)} = \mathbf{y}^{(0)}$*. The two dynamical systems in Eqs. (3.1) and (3.2) to constraint in Eq. (3.3) are penetrated of the* $(2k_1 : 2k_2)$*-type for time* $t \in [t_{m_1}, t_{m_2}]$ *if and only if*

(i) *for* $\mathbf{y}^{(\alpha)}_{m} = \mathbf{y}^{(0)}_{m}$ *and* $\mathbf{y}^{(\alpha)} \in \boldsymbol{\Omega}_{\alpha}$ *(*$\alpha \in \{1,2\}$*) at time* $t = t_m \in [t_{m_1}, t_{m_2}]$

$$\varphi(\mathbf{y}^{(\alpha)}_{m\pm}, t_{m\pm}, \boldsymbol{\lambda}) = \varphi(\mathbf{y}^{(0)}_{m}, t_m, \boldsymbol{\lambda}) = 0 \tag{3.90}$$

(ii) *at time* $t = t_m \in (t_{m_1}, t_{m_2})$*,* $\mathbf{y}^{(\alpha)}_{m-} = \mathbf{y}^{(0)}_{m} = \mathbf{y}^{(\beta)}_{m+}$

$$\varphi^{(s_\alpha)}(\mathbf{y}^{(\alpha)}_{m-}, t_{m-}, \boldsymbol{\lambda}) = 0 \quad (s_\alpha = 1, 2, \cdots, 2k_\alpha),$$
$$(-1)^{\alpha}\varphi^{(2k_\alpha+1)}(\mathbf{y}^{(\alpha)}_{m-}, t_{m-}, \boldsymbol{\lambda}) > 0;$$
$$\varphi^{(s_\beta)}(\mathbf{y}^{(\beta)}_{m+}, t_{m+}, \boldsymbol{\lambda}) = 0 \quad (s_\beta = 1, 2, \cdots, 2k_\beta),$$
$$(-1)^{\beta}\varphi^{(2k_\beta+1)}(\mathbf{y}^{(\beta)}_{m+}, t_{m+}, \boldsymbol{\lambda}) < 0 \text{ for } \beta \in \{1,2\} \text{ and } \alpha \neq \beta; \tag{3.91}$$

(iii) *with a* $(2k_1 : 2k_2)$*-synchronization at time* $t = t_{m_i}$*,* $\mathbf{y}^{(\alpha)}_{m_i-} = \mathbf{y}^{(0)}_{m_i} = \mathbf{y}^{(\beta)}_{m_i\pm}$ *(*$i \in \{1,2\}$*)*

$$\varphi^{(s_\alpha)}(\mathbf{y}^{(\alpha)}_{m_i-}, t_{m_i-}, \boldsymbol{\lambda}) = 0 \quad (s_\alpha = 1, 2, \cdots, 2k_\alpha),$$
$$(-1)^{\alpha}\varphi^{(2k_\alpha+1)}(\mathbf{y}^{(\alpha)}_{m_i-}, t_{m_i-}, \boldsymbol{\lambda}) > 0,$$
$$\varphi^{(s_\beta)}(\mathbf{y}^{(\beta)}_{m_i\pm}, t_{m_i\pm}, \boldsymbol{\lambda}) = 0 \quad (s_\beta = 1, 2, \cdots, 2k_\beta + 1),$$
$$(-1)^{\beta}\varphi^{(2k_\beta+2)}(\mathbf{y}^{(\beta)}_{m_i\pm}, t_{m_i\pm}, \boldsymbol{\lambda}) < 0 \text{ for } \beta \in \{1,2\} \text{ and } \alpha \neq \beta, \tag{3.92}$$

or with a $(2k_1 : 2k_2)$ *desynchronization at time* $t = t_{m_i}$*,* $\mathbf{y}^{(\alpha)}_{m_i\mp} = \mathbf{y}^{(0)}_{m_i} = \mathbf{y}^{(\beta)}_{m_i+}$ *(*$i \in \{1,2\}$*),*

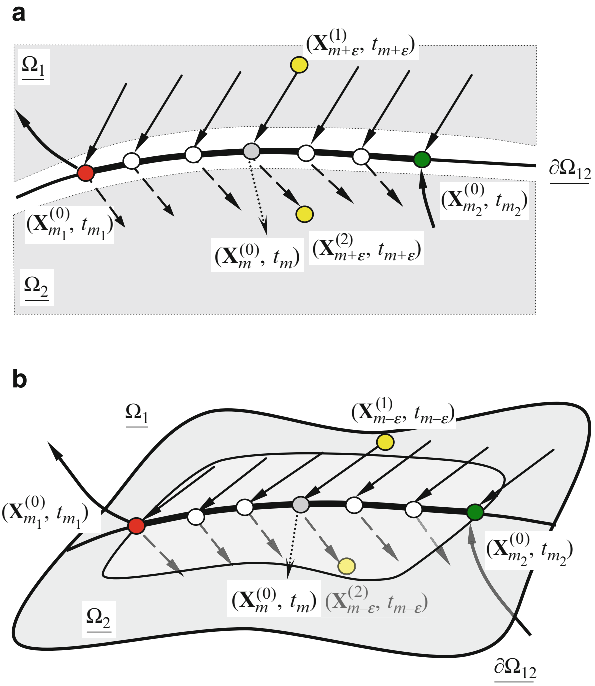

Fig. 3.11 (**a**) A cross-section view and (**b**) a three-dimensional view of the penetration of resultant flows in vicinity of the constraint boundary $\partial\Omega_{12}$ in $(n_r + n_s)$-dimensional state space. On the constraint boundary, any point for penetration is expressed by $(\mathbf{y}^{(0)}_m, t_m)$. In two domains, the resultant flows in the vicinity of the constraint boundary are expressed by $(\mathbf{y}^{(\alpha)}_{m-\varepsilon}, t_{m-\varepsilon})$ $(\alpha = 1, 2)$. The onset and vanishing points are $(\mathbf{y}^{(0)}_{m_1}, t_{m_1})$ and $(\mathbf{y}^{(0)}_{m_2}, t_{m_2})$ with *red* and *blue circular symbols*

$$
\begin{aligned}
&\varphi^{(s_\alpha)}(\mathbf{y}^{(\alpha)}_{m_i\mp}, t_{m_i\mp}, \boldsymbol{\lambda}) = 0 \quad (s_\alpha = 1, 2, \cdots, 2k_\alpha + 1),\\
&(-1)^\alpha \varphi^{(2k_\alpha+1)}(\mathbf{y}^{(\alpha)}_{m_i\mp}, t_{m_i\mp}, \boldsymbol{\lambda}) < 0,\\
&\varphi^{(s_\beta)}(\mathbf{y}^{(\beta)}_{m+}, t_{m+}, \boldsymbol{\lambda}) = 0 \quad (s_\beta = 1, 2, \cdots, 2k_\beta),\\
&(-1)^\beta \varphi^{(2k_\beta+1)}(\mathbf{y}^{(\beta)}_{m+}, t_{m+}, \boldsymbol{\lambda}) < 0 \text{ for } \beta \in \{1,2\} \text{ and } \alpha \neq \beta,
\end{aligned} \tag{3.93}
$$

or with a $(2k_\beta : 2k_\alpha)$*-penetration at time* $t = t_{m_i}$, $\mathbf{y}^{(\alpha)}_{m_i\mp} = \mathbf{y}^{(0)}_{m_i} = \mathbf{y}^{(\beta)}_{m_i\pm} (i \in \{1, 2\})$,

$$
\begin{aligned}
&\varphi^{(s_\alpha)}(\mathbf{y}^{(\alpha)}_{m_i\mp}, t_{m_i\mp}, \boldsymbol{\lambda}) = 0 \quad (s_\alpha = 1, 2, \cdots, 2k_\alpha + 1),\\
&(-1)^\alpha \varphi^{(2k_\alpha+1)}(\mathbf{y}^{(\alpha)}_{m_i\mp}, t_{m_i\mp}, \boldsymbol{\lambda}) < 0,\\
&\varphi^{(s_\beta)}(\mathbf{y}^{(\beta)}_{m_i\pm}, t_{m_i\pm}, \boldsymbol{\lambda}) = 0 \quad (s_\beta = 1, 2, \cdots, 2k_\beta + 1),\\
&(-1)^\beta \varphi^{(2k_\beta+2)}(\mathbf{y}^{(\beta)}_{m_i\pm}, t_{m_i\pm}, \boldsymbol{\lambda}) < 0 \text{ for } \beta \in \{1,2\} \text{ and } \alpha \neq \beta.
\end{aligned} \tag{3.94}
$$

Proof Using Definitions 3.21, 3.23, 3.24, and 3.26, the proof of this theorem is similar to the proof of Theorem 3.4. □

The penetration of the two dynamical systems to a specific constraint is sketched in Fig. 3.11. The *onset and vanishing* conditions of the $(2k_\alpha : 2k_\beta)$-penetration of the $t \in [t_{m_1}, t_{m_2}]$ to a specific constraint are the *vanishing and onset* conditions of the $(2k_\alpha : 2k_\beta)$-synchronization, the $(2k_\alpha : 2k_\beta)$-desynchronization, and the $(2k_\beta : 2k_\alpha)$ -penetration, respectively. On the constraint boundary, any point for penetration is expressed by $(\mathbf{y}_m^{(0)}, t_m)$. In two domains, the incoming and output resultant flows in the vicinity of the constraint boundary are expressed by $(\mathbf{y}_{m-\varepsilon}^{(\alpha)}, t_{m-\varepsilon})$ and $(\mathbf{y}_{m+\varepsilon}^{(\beta)}, t_{m+\varepsilon})$ $(\alpha, \beta \in \{1, 2\}$ and $\alpha \neq \beta)$. The *onset and vanishing* points are $(\mathbf{y}_{m_1}^{(0)}, t_{m_1})$ and $(\mathbf{y}_{m_2}^{(0)}, t_{m_2})$ with red and blue circular symbols.

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Chapter 4
Multiple Constraints Synchronization

In this chapter, the synchronization of two dynamical systems to multiple constraints will be discussed. As in Luo [1], the synchronicity of two dynamical systems with multiple constraints can be presented. The mathematical description of the synchronicity of two dynamical systems to multiple constraints will be given, and the corresponding necessary and sufficient conditions for the synchronicity of two dynamical systems to the constraints are presented.

4.1 Synchronicity to Multiple Constraints

The ε-domain in the vicinity of the intersected, constraint boundary will be defined through the ε-domain of the jth constraint boundary. Based on such ε-domain and the intersected constraint boundary, the synchronicity of two dynamical systems to multiple constraints will be discussed.

Definition 4.1 For $\mathbf{y}_{m\pm}^{(\alpha_j,j)} \in \Omega_{(\alpha_j,j)}$ $(\alpha_j \in \mathscr{I}$ and $j \in \mathscr{L}$ with $\mathscr{I} = \{1,2\}$ and $\mathscr{L} = \{1,2,\cdots,l\})$ and $\mathbf{y}_m^{(0,j)} \in \partial\Omega_{12(j)}$ at time t_m, $\mathbf{y}_{m\pm}^{(\alpha_j,j)} = \mathbf{y}_m^{(0,j)}$. For any small $\varepsilon>0$, there is a time interval $[t_{m-\varepsilon}, t_m)$ or $(t_m, t_{m+\varepsilon}]$. The neighborhood of the jth constraint edge is defined as

$$\begin{aligned}
\Omega_{(\alpha_j,j)}^{-\varepsilon} &= \{\mathbf{y}^{(\alpha_j,j)} \big| \|\mathbf{y}^{(\alpha_j,j)}(t) - \mathbf{y}_m^{(0,j)}\| \le \delta_{(\alpha_j,j)}, \delta_{(\alpha_j,j)}>0, t \in [t_{m-\varepsilon}, t_m)\},\\
\Omega_{(\alpha_j,j)}^{+\varepsilon} &= \{\mathbf{y}^{(\alpha_j,j)} \big| \|\mathbf{y}^{(\alpha_j,j)}(t) - \mathbf{y}_m^{(0,j)}\| \le \delta_{(\alpha_j,j)}, \delta_{(\alpha_j,j)}>0, t \in (t_m, t_{m+\varepsilon}]\}.
\end{aligned} \tag{4.1}$$

The subdomains and the intersected edge are defined as

$$\Omega_{\boldsymbol{\alpha}} \equiv \Omega_{\alpha_1\alpha_2\cdots\alpha_l} = \cap_{j=1}^{l}\Omega_{(\alpha_j,\, j)} \text{ and } \partial\Omega_{(12,\mathbf{j})} \equiv \partial\Omega_{(12,12\cdots l)} = \cap_{j=1}^{l}\partial\Omega_{(\alpha_j\beta_j,j)} \tag{4.2}$$

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For $\mathbf{y}_{m\pm}^{(\boldsymbol{\alpha})} \in \Omega_{\boldsymbol{\alpha}}$ ($\boldsymbol{\alpha} = (\alpha_1\alpha_2\cdots\alpha_l)$, $\alpha_j \in \mathscr{I}$ and $j \in \mathscr{L}$) and $\mathbf{y}_m^{(0)} \in \partial\Omega_{(12,\mathbf{j})}$ ($\mathbf{j} = (12\cdots l)$) at time t_m, $\mathbf{y}_m^{(\boldsymbol{\alpha})} = \mathbf{y}_m^{(0)}$. For any small $\varepsilon>0$, there is a time interval $[t_{m-\varepsilon}, t_m)$ or $(t_m, t_{m+\varepsilon}]$. The neighborhood of the intersected constraint boundary $\partial\Omega_{(12,\mathbf{j})}$ is defined as

$$\left.\begin{aligned}\Omega_{\boldsymbol{\alpha}}^{-\varepsilon} &= \{\mathbf{y}^{(\boldsymbol{\alpha})} \,\big|\, ||\mathbf{y}^{(\boldsymbol{\alpha})}(t) - \mathbf{y}_m^{(0)}|| \le \delta, \delta>0, t \in [t_{m-\varepsilon}, t_m)\},\\ \Omega_{\boldsymbol{\alpha}}^{+\varepsilon} &= \{\mathbf{y}^{(\boldsymbol{\alpha})} \,\big|\, ||\mathbf{y}^{(\boldsymbol{\alpha})}(t) - \mathbf{y}_m^{(0)}|| \le \delta, \delta>0, t \in (t_m, t_{m+\varepsilon}]\}.\end{aligned}\right\} \tag{4.3}$$

where $\delta = \min_{j\in\mathscr{L},\alpha_j\in\mathscr{I}} (\delta_{(\alpha_j,j)})$ with $\mathscr{L} = \{1,2,\cdots,l\}$ and $\mathscr{I} = \{1,2\}$.

Definition 4.2 Three index sets are defined as

$$\mathscr{L} = \cup_{i=1}^{3} \mathscr{L}_i \quad \text{and} \quad \mathscr{L}_i \cap \mathscr{L}_j = \{\varnothing\}\ (i,j \in \{1,2,3\}) \tag{4.4}$$

$$\mathscr{L}_i = \{\varnothing, k_1^{(i)}, k_2^{(i)}, \cdots, k_{l_i}^{(i)}\} \subseteq \mathscr{L} \cup \{\varnothing\} \quad \text{and} \quad l_1 + l_2 + l_3 = l \tag{4.5}$$

Definition 4.3 Consider two dynamical systems in Eqs. (3.1) and (3.2) with constraints in Eq. (3.4). For $\mathbf{y}_{m\pm}^{(\alpha_j,j)} \in \Omega_{(\alpha_j,j)}$ ($\alpha_j \in \mathscr{I}$ and $j \in \mathscr{L}$ with $\mathscr{I} = \{1,2\}$ and $\mathscr{L} = \{1,2,\cdots,l\}$) and $\mathbf{y}_m^{(0,j)} \in \partial\Omega_{(12,j)}$ at time t_m, $\mathbf{y}_{m\pm}^{(\alpha_j,j)} = \mathbf{y}_m^{(0,j)}$. For any small $\varepsilon > 0$, there is a time interval $[t_{m-\varepsilon}, t_m)$ or $(t_m, t_{m+\varepsilon}]$. The two systems in Eqs. (3.1) and (3.2) with constraints in Eq. (3.4) are called an l_1-dimensional synchronization, l_2-dimensional desynchronization, and l_3-dimensional penetration for time $t_m \in [t_{m_1}, t_{m_2}]$

(i) if for α_{j_1} = 1,2 and $j_1 \in \mathscr{L}_1$

$$\left.\begin{aligned}&\varphi_{j_1}(\mathbf{y}_{m-}^{(\alpha_{j_1},j_1)}, t_{m-}, \boldsymbol{\lambda}_{j_1}) = \varphi_{j_1}(\mathbf{y}_m^{(0,j_1)}, t_m, \boldsymbol{\lambda}_{j_1}) = 0,\\ &(-1)^{\alpha_{j_1}}[\varphi_{j_1}(\mathbf{y}_{m-\varepsilon}^{(\alpha_{j_1},j_1)}, t_{m-\varepsilon}, \boldsymbol{\lambda}_{j_1}) - \varphi_{j_1}(\mathbf{y}_{m-}^{(\alpha_{j_1},j_1)}, t_{m-}, \boldsymbol{\lambda}_{j_1})]<0,\end{aligned}\right\} \tag{4.6}$$

(ii) if for α_{j_2} = 1,2 and $j_2 \in \mathscr{L}_2$

$$\left.\begin{aligned}&\varphi_{j_2}(\mathbf{y}_{m+}^{(\alpha_{j_2},j_2)}, t_{m+}, \boldsymbol{\lambda}_{j_2}) = \varphi_{j_2}(\mathbf{y}_m^{(0,j_2)}, t_m, \boldsymbol{\lambda}_{j_2}) = 0,\\ &(-1)^{\alpha_{j_2}}[\varphi_{j_2}(\mathbf{y}_{m+\varepsilon}^{(\alpha_{j_2},j_2)}, t_{m+\varepsilon}, \boldsymbol{\lambda}_{j_2}) - \varphi_{j_2}(\mathbf{y}_{m+}^{(\alpha_{j_2},j_2)}, t_{m+}, \boldsymbol{\lambda}_{j_2})]<0,\end{aligned}\right\} \tag{4.7}$$

(iii) if for $\alpha_{j_3}, \beta_{j_3} \in \{1,2\}$, $\alpha_{j_3} \neq \beta_{j_3}; j_3 \in \mathscr{L}_3$

$$\left.\begin{aligned}&\varphi_{j_3}(\mathbf{y}_{m-}^{(\alpha_{j_3},j_3)}, t_{m-}, \boldsymbol{\lambda}_{j_3}) = \varphi_{j_3}(\mathbf{y}_m^{(0,j_3)}, t_m, \boldsymbol{\lambda}_{j_3}) = 0,\\ &(-1)^{\alpha_{j_3}}[\varphi_{j_3}(\mathbf{y}_{m-\varepsilon}^{(\alpha_{j_3},j_3)}, t_{m-\varepsilon}, \boldsymbol{\lambda}_{j_3}) - \varphi_{j_3}(\mathbf{y}_{m-}^{(\alpha_{j_3},j_3)}, t_{m-}, \boldsymbol{\lambda}_{j_3})]<0,\end{aligned}\right\} \tag{4.8}$$

$$\left.\begin{aligned}&\varphi_{j_3}(\mathbf{y}_{m+}^{(\beta_{j_3},j_3)},t_{m+},\boldsymbol{\lambda}_{j_3})=\varphi_{j_3}(\mathbf{y}_m^{(0,j_3)},t_m,\boldsymbol{\lambda}_{j_3})=0,\\&(-1)^{\beta_{j_3}}[\varphi_{j_3}(\mathbf{y}_{m+\varepsilon}^{(\beta_{j_3},j_3)},t_{m+\varepsilon},\boldsymbol{\lambda}_{j_3})-\varphi_{j_3}(\mathbf{y}_{m+}^{(\beta_{j_3},j_3)},t_{m+},\boldsymbol{\lambda}_{j_3})]<0.\end{aligned}\right\}\tag{4.9}$$

From the previous definition, among the l-constraints in Eq. (3.4), (i) there are l_1-constraints to make the two dynamical systems be synchronized at the corresponding constraints; (ii) there are l_2-constraints to make the two dynamical systems be desynchronized at the corresponding constraints; and (iii) there are l_3-constraints to make the two dynamical systems be penetrated at the corresponding constraints. If $l_2 = l_3 = 0$ and $l_1 = l$, the two dynamical systems to all the l-constraints are synchronized. If $l_3 = 0$ and $l_1 + l_2 = l$, the two dynamical systems to all the l-constraints are to be synchronized with l_1-constraints and desynchronized with l_2-constraints. If $l_1 = 0$ and $l_2 + l_3 = l$, the two dynamical systems to all the l-constraints are to be desynchronized with l_2-constraints and to be penetrated with l_3-constraints. If $l_2 = 0$ and $l_1 + l_3 = l$, the two dynamical systems to all the l-constraints are to be synchronized with l_1-constraints and to be penetrated with l_3-constraints. For the two cases, the two dynamical systems cannot be synchronized any more for all the l-constraints. If one of three types of synchronicity has changed the current state, the synchronicity of the two dynamical systems will be changed. The three special cases are useful. Therefore, three definitions for the three special cases will be given as follows:

Definition 4.4 Consider two dynamical systems in Eqs. (3.1) and (3.2) with constraints in Eq. (3.4). For $\mathbf{y}_{m-}^{(\alpha_j,j)} \in \Omega_{(\alpha_j,j)}$ ($\alpha_j \in \mathscr{I}$ and $j \in \mathscr{L}$ with $\mathscr{I} = \{1,2\}$ and $\mathscr{L} = \{1,2,\cdots,l\}$) and $\mathbf{y}_m^{(0,j)} \in \partial\Omega_{(12,j)}$ at time t_m, $\mathbf{y}_{m-}^{(\alpha_j,j)} = \mathbf{y}_m^{(0,j)}$. For any small $\varepsilon > 0$, there is a time interval $[t_{m-\varepsilon}, t_m)$. The two dynamical systems in Eqs. (3.1) and (3.2) with constraints in Eq. (3.4) are called an l-dimensional synchronization for time $t_m \in [t_{m_1}, t_{m_2}]$ if for $\alpha_j = 1,2$ and $j \in \mathscr{L}$

$$\left.\begin{aligned}&\varphi_j(\mathbf{y}_{m-}^{(\alpha_j,j)},t_{m-},\boldsymbol{\lambda}_j)=\varphi_j(\mathbf{y}_m^{(0,j)},t_m,\boldsymbol{\lambda}_j)=0,\\&(-1)^{\alpha_j}[\varphi_j(\mathbf{y}_{m-\varepsilon}^{(\alpha_j,j)},t_{m-\varepsilon},\boldsymbol{\lambda}_j)-\varphi_j(\mathbf{y}_{m-}^{(\alpha_j,j)},t_{m-},\boldsymbol{\lambda}_j)]<0.\end{aligned}\right\}\tag{4.10}$$

Definition 4.5 Consider two dynamical systems in Eqs. (3.1) and (3.2) with constraints in Eq. (3.4). For $\mathbf{y}_{m+}^{(\alpha_j,j)} \in \Omega_{(\alpha_j,j)}$ ($\alpha_j \in \mathscr{I}$ and $j \in \mathscr{L}$ with $\mathscr{I} = \{1,2\}$ and $\mathscr{L} = \{1,2,\cdots,l\}$) and $\mathbf{y}_m^{(0,j)} \in \partial\Omega_{(12,j)}$ at time t_m, $\mathbf{y}_{m+}^{(\alpha_j,j)} = \mathbf{y}_m^{(0,j)}$. For any small $\varepsilon > 0$, there is a time interval $(t_m, t_{m+\varepsilon}]$. The two dynamical systems in Eqs. (3.1) and (3.2) with constraints in Eq. (3.4) are called an l-dimensional desynchronization for time $t_m \in [t_{m_1}, t_{m_2}]$ if for $\alpha_j = 1,2$ and $j \in \mathscr{L}$

$$\left.\begin{aligned}&\varphi_j(\mathbf{y}_{m+}^{(\alpha_j,j)},t_{m+},\boldsymbol{\lambda}_j)=\varphi_j(\mathbf{y}_m^{(0,j)},t_m,\boldsymbol{\lambda}_j)=0,\\&(-1)^{\alpha_j}[\varphi_j(\mathbf{y}_{m+\varepsilon}^{(\alpha_j,j)},t_{m+\varepsilon},\boldsymbol{\lambda}_j)-\varphi_j(\mathbf{y}_{m+}^{(\alpha_j,j)},t_{m+},\boldsymbol{\lambda}_j)]<0.\end{aligned}\right\}\tag{4.11}$$

Definition 4.6 Consider two dynamical systems in Eqs. (3.1) and (3.2) with constraints in Eq. (3.4). For $\mathbf{y}_{m\pm}^{(\alpha_j,j)} \in \Omega_{(\alpha_j,j)}$ ($\alpha_j \in \mathscr{I}$ and $j \in \mathscr{L}$ with $\mathscr{I} = \{1,2\}$ and $\mathscr{L} = \{1,2,\cdots,l\}$) and $\mathbf{y}_m^{(0,j)} \in \partial\Omega_{(12,j)}$ at time t_m, $\mathbf{y}_{m\pm}^{(\alpha_j,j)} = \mathbf{y}_m^{(0,j)}$. For any small $\varepsilon > 0$, there is a time interval $[t_{m-\varepsilon}, t_m)$ or $(t_m, t_{m+\varepsilon}]$. The two dynamical systems in Eqs. (3.1) and (3.2) with constraints in Eq. (3.4) are called an l-dimensional penetration for time $t_m \in [t_{m_1}, t_{m_2}]$ if for $\alpha_j,\beta_j \in \mathscr{I}$, $j \in \mathscr{L}$ and $\beta_j \neq \alpha_j$

$$\left.\begin{aligned} &\varphi_j(\mathbf{y}_{m-}^{(\alpha_j,j)}, t_{m-}, \boldsymbol{\lambda}_j) = \varphi_j(\mathbf{y}_m^{(0,j)}, t_m, \boldsymbol{\lambda}_j) = 0,\\ &(-1)^{\alpha_j}[\varphi_j(\mathbf{y}_{m-\varepsilon}^{(\alpha_j,j)}, t_{m-\varepsilon}, \boldsymbol{\lambda}_j) - \varphi_j(\mathbf{y}_{m-}^{(\alpha_j,j)}, t_{m-}, \boldsymbol{\lambda}_j)] < 0, \end{aligned}\right\} \tag{4.12}$$

$$\left.\begin{aligned} &\varphi_j(\mathbf{y}_{m+}^{(\beta_j,j)}, t_{m+}, \boldsymbol{\lambda}_j) = \varphi_j(\mathbf{y}_m^{(0,j)}, t_m, \boldsymbol{\lambda}_j) = 0,\\ &(-1)^{\beta_j}[\varphi_j(\mathbf{y}_{m+\varepsilon}^{(\beta_j,j)}, t_{m+\varepsilon}, \boldsymbol{\lambda}_j) - \varphi_j(\mathbf{y}_{m+}^{(\beta_j,j)}, t_{m+}, \boldsymbol{\lambda}_j)] < 0. \end{aligned}\right\} \tag{4.13}$$

4.2 Singularity to Constraints

As discussed in the synchronization of two dynamical systems to the single constraint, the singularity for a flow of the two dynamical systems in Eqs. (3.1) and (3.2) to one of the constraints in Eq. (3.4) can be described. The tangency of a resultant flow to one of l-constraint boundaries is presented first, and then the vanishing and onset of the synchronization of two dynamical systems to the jth constraint boundary of the l-constraint boundaries will be presented.

Definition 4.7 Consider two dynamical systems in Eqs. (3.1) and (3.2) with constraints in Eq. (3.4). For $\mathbf{y}_{m\pm}^{(\alpha_j,j)} \in \Omega_{(\alpha_j,j)}$ ($\alpha_j \in \mathscr{I}$ and $j \in \mathscr{L}$ with $\mathscr{I} = \{1,2\}$and $\mathscr{L} = \{1,2,\cdots,l\}$) and $\mathbf{y}_m^{(0,j)} \in \partial\Omega_{(12,j)}$ at time t_m, $\mathbf{y}_{m\pm}^{(\alpha_j,j)} = \mathbf{y}_m^{(0,j)}$. For any small $\varepsilon > 0$, there is a time interval $[t_{m-\varepsilon}, t_m)$ or $(t_m, t_{m+\varepsilon}]$. At $\mathbf{y}^{(\alpha_j,j)} \in \Omega_{(\alpha_j,j)}^{\pm\varepsilon}$ for time $t \in [t_{m-\varepsilon}, t_m)$ or $(t_m, t_{m+\varepsilon}]$, the constraint function $\varphi_j(\mathbf{y}^{(\alpha_j,j)}, t, \boldsymbol{\lambda}_j)$ is $C^{r_{\alpha_j}}$ continuous and $|\varphi_j^{(r_{\alpha_j}+1)}(\mathbf{y}^{(\alpha_j,j)}, t, \boldsymbol{\lambda}_j)| < \infty$ ($r_{\alpha_j} \geq 2$). A flow of the resultant system for two dynamical systems in Eqs. (3.1) and (3.2) with l-constraints in Eq. (3.4) is called *to be tangential to* the jth constraint boundary for time $t_m \in [t_{m_1}, t_{m_2}]$ if for $j \in \mathscr{L}$ and $\alpha_j \in \mathscr{I}$

$$\left.\begin{aligned} &\varphi_j(\mathbf{y}_{m\mp}^{(\alpha_j,j)}, t_{m\mp}, \boldsymbol{\lambda}_j) = \varphi_j(\mathbf{y}_m^{(0,j)}, t_m, \boldsymbol{\lambda}_j) = 0,\\ &\varphi_j^{(1)}(\mathbf{y}_{m\mp}^{(\alpha_j,j)}, t_{m\mp}, \boldsymbol{\lambda}_j) = 0,\\ &(-1)^{\alpha_j}[\varphi_j(\mathbf{y}_{m\mp\varepsilon}^{(\alpha_j,j)}, t_{m\mp\varepsilon}, \boldsymbol{\lambda}_j) - \varphi_j(\mathbf{y}_{m\mp}^{(\alpha_j,j)}, t_{m\mp}, \boldsymbol{\lambda}_j)] < 0. \end{aligned}\right\} \tag{4.14}$$

Definition 4.8 Consider two dynamical systems in Eqs. (3.1) and (3.2) with constraints in Eq. (3.4). For $\mathbf{y}_{m\pm}^{(\alpha_j,j)} \in \Omega_{(\alpha_j,j)}$ ($\alpha_j \in \mathscr{I}$ and $j \in \mathscr{L}$ with $\mathscr{I} = \{1,2\}$ and $\mathscr{L} = \{1,2,\cdots,l\}$) and $\mathbf{y}_m^{(0,j)} \in \partial\Omega_{(12,j)}$ at time t_m, $\mathbf{y}_{m\pm}^{(\alpha_j,j)} = \mathbf{y}_m^{(0,j)}$. For any small $\varepsilon > 0$, there is a time interval $[t_{m-\varepsilon}, t_m)$ or $(t_m, t_{m+\varepsilon}]$. At $\mathbf{y}^{(\alpha_j,j)} \in \Omega_{(\alpha_j,j)}^{\pm\varepsilon}$ for time $t \in [t_{m-\varepsilon}, t_m)$ or $(t_m, t_{m+\varepsilon}]$, the constraint function $\varphi_j(\mathbf{y}^{(\alpha_j,j)}, t, \boldsymbol{\lambda}_j)$ is $C^{r_{\alpha_j}}$ continuous and $|\varphi_j^{(r_{\alpha_j}+1)}(\mathbf{y}^{(\alpha_j,j)},\ t, \boldsymbol{\lambda}_j)| < \infty$ ($r_{\alpha_j} \geq 2$).

(i) The synchronization of the two dynamical systems in Eqs. (3.1) and (3.2) with the jth constraint in Eq. (3.4) is called to be *vanishing* to form a penetration from $\Omega_{(\alpha_j,j)}$ to $\Omega_{(\beta_j,j)}$ on the jth constraint boundary at time t_m if for $\alpha_j, \beta_j \in \mathscr{I}$ and $\alpha_j \neq \beta_j$ with $j \in \mathscr{L}$

$$\left.\begin{aligned}
&\varphi_j(\mathbf{y}_{m-}^{(\alpha_j,j)}, t_{m-}, \boldsymbol{\lambda}_j) = \varphi_j(\mathbf{y}_{m\mp}^{(\beta_j,j)}, t_{m\mp}, \boldsymbol{\lambda}_j) = \varphi_j(\mathbf{y}_m^{(0,j)}, t_m, \boldsymbol{\lambda}_j) = 0,\\
&\varphi_j^{(1)}(\mathbf{y}_{m\mp}^{(\beta_j,j)}, t_{m\mp}, \boldsymbol{\lambda}_j) = 0,\\
&(-1)^{\alpha_j}[\varphi_j(\mathbf{y}_{m-\varepsilon}^{(\alpha_j,j)}, t_{m-\varepsilon}, \boldsymbol{\lambda}_j) - \varphi_j(\mathbf{y}_{m-}^{(\alpha_j,j)}, t_{m-}, \boldsymbol{\lambda}_j)] < 0,\\
&(-1)^{\beta_j}[\varphi_j(\mathbf{y}_{m\mp\varepsilon}^{(\beta_j,j)}, t_{m\mp\varepsilon}, \boldsymbol{\lambda}_j) - \varphi_j(\mathbf{y}_{m\mp}^{(\beta_j,j)}, t_{m\mp}, \boldsymbol{\lambda}_j)] < 0.
\end{aligned}\right\} \tag{4.15}$$

(ii) The synchronization of the two dynamical systems in Eqs. (3.1) and (3.2) with the jth constraint in Eq. (3.4) is called to be *onset* from the penetration on the jth constraint boundary at time t_m if for $\alpha_j, \beta_j \in \mathscr{I}$ and $\alpha_j \neq \beta_j$ with $j \in \mathscr{L}$

$$\left.\begin{aligned}
&\varphi_j(\mathbf{y}_{m-}^{(\alpha_j,j)}, t_{m-}, \boldsymbol{\lambda}_j) = \varphi_j(\mathbf{y}_{m\pm}^{(\beta_j,j)}, t_{m\pm}, \boldsymbol{\lambda}_j) = \varphi_j(\mathbf{y}_m^{(0,j)}, t_m, \boldsymbol{\lambda}_j) = 0,\\
&\varphi_j^{(1)}(\mathbf{y}_{m\pm}^{(\beta_j,j)}, t_{m\pm}, \boldsymbol{\lambda}_j) = 0,\\
&(-1)^{\alpha_j}[\varphi_j(\mathbf{y}_{m-\varepsilon}^{(\alpha_j,j)}, t_{m-\varepsilon}, \boldsymbol{\lambda}_j) - \varphi_j(\mathbf{y}_{m-}^{(\alpha_j,j)}, t_{m-}, \boldsymbol{\lambda}_j)] < 0,\\
&(-1)^{\beta_j}[\varphi_j(\mathbf{y}_{m\pm\varepsilon}^{(\beta_j,j)}, t_{m\pm\varepsilon}, \boldsymbol{\lambda}_j) - \varphi_j(\mathbf{y}_{m\pm}^{(\beta_j,j)}, t_{m\pm}, \boldsymbol{\lambda}_j)] < 0.
\end{aligned}\right\} \tag{4.16}$$

Definition 4.9 Consider two dynamical systems in Eqs. (3.1) and (3.2) with constraints in Eq. (3.4). For $\mathbf{y}_{m\pm}^{(\alpha_j,j)} \in \Omega_{(\alpha_j,j)}$ ($\alpha_j \in \mathscr{I}$ and $j \in \mathscr{L}$ with $\mathscr{I} = \{1,2\}$ and $\mathscr{L} = \{1,2,\cdots,l\}$) and $\mathbf{y}_m^{(0,j)} \in \partial\Omega_{(12,j)}$ at time t_m, $\mathbf{y}_{m\pm}^{(\alpha_j,j)} = \mathbf{y}_m^{(0,j)}$. For any small $\varepsilon > 0$, there is a time interval $[t_{m-\varepsilon}, t_m)$ or $(t_m, t_{m+\varepsilon}]$. At $\mathbf{y}^{(\alpha_j,j)} \in \Omega_{(\alpha_j,j)}^{\pm\varepsilon}$ for time $t \in [t_{m-\varepsilon}, t_m)$ or $(t_m, t_{m+\varepsilon}]$, the constraint function $\varphi_j(\mathbf{y}^{(\alpha_j,j)}, t, \boldsymbol{\lambda}_j)$ is $C^{r_{\alpha_j}}$ continuous and $|\varphi_j^{(r_{\alpha_j}+1)}(\mathbf{y}^{(\alpha_j,j)}, t, \boldsymbol{\lambda}_j)| < \infty$ ($r_{\alpha_j} \geq 2$).

(i) The synchronization of the two dynamical systems in Eqs. (3.1) and (3.2) with the jth constraint in Eq. (3.4) is called to be *onset* from the desynchronization on the jth constraint boundary at time t_m if for $\alpha_j, \beta_j \in \mathscr{I}$ and $\alpha_j \neq \beta_j$ with $j \in \mathscr{L}$

$$\left.\begin{array}{l}\varphi_j(\mathbf{y}_{m\pm}^{(\alpha_j,j)},t_{m\pm},\boldsymbol{\lambda}_j)=\varphi_j(\mathbf{y}_{m\pm}^{(\beta_j,j)},t_{m\pm},\boldsymbol{\lambda}_j)=\varphi_j(\mathbf{y}_m^{(0,j)},t_m,\boldsymbol{\lambda}_j)=0,\\ \varphi_j^{(1)}(\mathbf{y}_{m\pm}^{(\alpha_j,j)},t_{m\pm},\boldsymbol{\lambda}_j)=\varphi_j^{(1)}(\mathbf{y}_{m\pm}^{(\beta_j,j)},t_{m\pm},\boldsymbol{\lambda}_j)=0,\\ (-1)^{\alpha_j}[\varphi_j(\mathbf{y}_{m\pm\varepsilon}^{(\alpha_j,j)},t_{m\pm\varepsilon},\boldsymbol{\lambda}_j)-\varphi_j(\mathbf{y}_{m\pm}^{(\alpha_j,j)},t_{m\pm},\boldsymbol{\lambda}_j)]<0,\\ (-1)^{\beta_j}[\varphi_j(\mathbf{y}_{m\pm\varepsilon}^{(\beta_j,j)},t_{m\pm\varepsilon},\boldsymbol{\lambda}_j)-\varphi_j(\mathbf{y}_{m\pm}^{(\beta_j,j)},t_{m\pm},\boldsymbol{\lambda}_j)]<0.\end{array}\right\} \quad (4.17)$$

(ii) The synchronization of the two dynamical systems in Eqs. (3.1) and (3.2) with the jth constraint in Eq. (3.4) is called to be *vanishing* to form the desynchronization on the jth constraint boundary at time t_m if for $\alpha_j, \beta_j \in \mathscr{I}$ and $\alpha_j \neq \beta_j$ with $j \in \mathscr{L}$

$$\left.\begin{array}{l}\varphi_j(\mathbf{y}_{m\mp}^{(\alpha_j,j)},t_{m\mp},\boldsymbol{\lambda}_j)=\varphi_j(\mathbf{y}_{m\mp}^{(\beta_j,j)},t_{m\mp},\boldsymbol{\lambda}_j)=\varphi_j(\mathbf{y}_m^{(0,j)},t_m,\boldsymbol{\lambda}_j)=0,\\ \varphi_j^{(1)}(\mathbf{y}_{m\mp}^{(\alpha_j,j)},t_{m\mp},\boldsymbol{\lambda}_j)=\varphi_j^{(1)}(\mathbf{y}_{m\mp}^{(\beta_j,j)},t_{m\mp},\boldsymbol{\lambda}_j)=0,\\ (-1)^{\alpha_j}[\varphi_j(\mathbf{y}_{m\mp\varepsilon}^{(\alpha_j,j)},t_{m\mp\varepsilon},\boldsymbol{\lambda}_j)-\varphi_j(\mathbf{y}_{m\mp}^{(\alpha_j,j)},t_{m\mp},\boldsymbol{\lambda}_j)]<0,\\ (-1)^{\beta_j}[\varphi_j(\mathbf{y}_{m\mp\varepsilon}^{(\beta_j,j)},t_{m\mp\varepsilon},\boldsymbol{\lambda}_j)-\varphi_j(\mathbf{y}_{m\mp}^{(\beta_j,j)},t_{m\mp},\boldsymbol{\lambda}_j)]<0.\end{array}\right\} \quad (4.18)$$

Definition 4.10 Consider two dynamical systems in Eqs. (3.1) and (3.2) with constraints in Eq. (3.4). For $\mathbf{y}_{m\pm}^{(\alpha_j,j)} \in \Omega_{(\alpha_j,j)}$ ($\alpha_j \in \mathscr{I}$ and $j \in \mathscr{L}$ with $\mathscr{I} = \{1,2\}$ and $\mathscr{L} = \{1,2,\cdots,l\}$) and $\mathbf{y}_m^{(0,j)} \in \partial\Omega_{(12,j)}$ at time t_m, $\mathbf{y}_{m\pm}^{(\alpha_j,j)} = \mathbf{y}_m^{(0,j)}$. For any small $\varepsilon > 0$, there is a time interval $[t_{m-\varepsilon}, t_m)$ or $(t_m, t_{m+\varepsilon}]$. At $\mathbf{y}^{(\alpha_j,j)} \in \Omega_{(\alpha_j,j)}^{\pm\varepsilon}$ for time $t \in [t_{m-\varepsilon}, t_m)$ or $(t_m, t_{m+\varepsilon}]$, the constraint function $\varphi_j(\mathbf{y}^{(\alpha_j,j)}, t, \boldsymbol{\lambda}_j)$ is $C^{r_{\alpha_j}}$ continuous and $|\varphi_j^{(r_{\alpha_j}+1)}(\mathbf{y}^{(\alpha_j,j)}, t, \boldsymbol{\lambda}_j)| < \infty$ ($r_{\alpha_j} \geq 2$).

(i) The desynchronization of the two dynamical systems in Eqs. (3.1) and (3.2) with the jth constraint in Eq. (3.4) is called to be *vanishing* to form a penetration on the jth constraint boundary at time t_m if for $\alpha_j, \beta_j \in \mathscr{I}$ and $\alpha_j \neq \beta_j$ with $j \in \mathscr{L}$

$$\left.\begin{array}{l}\varphi_j(\mathbf{y}_{m\pm}^{(\alpha_j,j)},t_{m\pm},\boldsymbol{\lambda}_j)=\varphi_j(\mathbf{y}_{m+}^{(\beta_j,j)},t_{m+},\boldsymbol{\lambda}_j)=\varphi_j(\mathbf{y}_m^{(0,j)},t_m,\boldsymbol{\lambda}_j)=0,\\ \varphi_j^{(1)}(\mathbf{y}_{m\pm}^{(\alpha_j,j)},t_{m\pm},\boldsymbol{\lambda}_j)=0,\\ (-1)^{\alpha_j}[\varphi_j(\mathbf{y}_{m\pm\varepsilon}^{(\alpha_j,j)},t_{m\pm\varepsilon},\boldsymbol{\lambda}_j)-\varphi_j(\mathbf{y}_{m\pm}^{(\alpha_j,j)},t_{m\pm},\boldsymbol{\lambda}_j)]<0,\\ (-1)^{\beta_j}[\varphi_j(\mathbf{y}_{m+\varepsilon}^{(\beta_j,j)},t_{m+\varepsilon},\boldsymbol{\lambda}_j)-\varphi_j(\mathbf{y}_{m+}^{(\beta_j,j)},t_{m+},\boldsymbol{\lambda}_j)]<0.\end{array}\right\} \quad (4.19)$$

(ii) The desynchronization of the two dynamical systems in Eqs. (3.1) and (3.2) with the jth constraint in Eq. (3.4) is called to be *onset* from the jth penetration flow on the jth constraint boundary at time t_m if for $\alpha_j, \beta_j \in \mathscr{I}$ and $\alpha_j \neq \beta_j$ with $j \in \mathscr{L}$

$$\left.\begin{aligned}
&\varphi_j(\mathbf{y}_{m\mp}^{(\alpha_j,j)},t_{m\mp},\boldsymbol{\lambda}_j)=\varphi_j(\mathbf{y}_{m+}^{(\beta_j,j)},t_{m+},\boldsymbol{\lambda}_j)=\varphi_j(\mathbf{y}_{m}^{(0,j)},t_m,\boldsymbol{\lambda}_j)=0,\\
&\varphi_j^{(1)}(\mathbf{y}_{m\mp}^{(\alpha_j,j)},t_{m\mp},\boldsymbol{\lambda}_j)=0,\\
&(-1)^{\alpha_j}[\varphi_j(\mathbf{y}_{m\mp\varepsilon}^{(\alpha_j,j)},t_{m\mp\varepsilon},\boldsymbol{\lambda}_j)-\varphi_j(\mathbf{y}_{m\mp}^{(\alpha_j,j)},t_{m\mp},\boldsymbol{\lambda}_j)]<0,\\
&(-1)^{\beta_j}[\varphi_j(\mathbf{y}_{m+\varepsilon}^{(\beta_j,j)},t_{m+\varepsilon},\boldsymbol{\lambda}_j)-\varphi_j(\mathbf{y}_{m+}^{(\beta_j,j)},t_{m+},\boldsymbol{\lambda}_j)]<0.
\end{aligned}\right\}\tag{4.20}$$

Definition 4.11 Consider two dynamical systems in Eqs. (3.1) and (3.2) with constraints in Eq. (3.4). For $\mathbf{y}_{m\pm}^{(\alpha_j,j)}\in\Omega_{(\alpha_j,j)}$ ($\alpha_j\in\mathscr{I}$ and $j\in\mathscr{L}$ with $\mathscr{I}=\{1,2\}$ and $\mathscr{L}=\{1,2,\cdots,l\}$) and $\mathbf{y}_m^{(0,j)}\in\partial\Omega_{(12,j)}$ at time t_m, $\mathbf{y}_{m\pm}^{(\alpha_j,j)}=\mathbf{y}_m^{(0,j)}$. For any small $\varepsilon>0$, there is a time interval $[t_{m-\varepsilon},t_m)$ or $(t_m,t_{m+\varepsilon}]$. At $\mathbf{y}^{(\alpha_j,j)}\in\Omega_{(\alpha_j,j)}^{\pm\varepsilon}$ for time $t\in[t_{m-\varepsilon},t_m)$ or $(t_m,t_{m+\varepsilon}]$, the constraint function $\varphi_j(\mathbf{y}^{(\alpha_j,j)},t,\boldsymbol{\lambda}_j)$ is $C^{r_{\alpha_j}}$ continuous and $|\varphi_j^{(r_{\alpha_j}+1)}(\mathbf{y}^{(\alpha_j,j)},t,\boldsymbol{\lambda}_j)|<\infty$ ($r_{\alpha_j}\geq 2$). The penetration of the two dynamical systems in Eqs. (3.1) and (3.2) with the jth constraint in Eq. (3.4) is called to be *switching* to form a new penetration on the jth constraint boundary at time t_m if for $\alpha_j,\beta_j\in\mathscr{I}$ and $\alpha_j\neq\beta_j$ with $j\in\mathscr{L}$

$$\left.\begin{aligned}
&\varphi_j(\mathbf{y}_{m\mp}^{(\alpha_j,j)},t_{m\mp},\boldsymbol{\lambda}_j)=\varphi_j(\mathbf{y}_{m\pm}^{(\beta_j,j)},t_{m\pm},\boldsymbol{\lambda}_j)=\varphi_j(\mathbf{y}_{m}^{(0,j)},t_m,\boldsymbol{\lambda}_j)=0,\\
&\varphi_j^{(1)}(\mathbf{y}_{m\mp}^{(\alpha_j,j)},t_{m\mp},\boldsymbol{\lambda}_j)=\varphi_j^{(1)}(\mathbf{y}_{m\pm}^{(\beta_j,j)},t_{m\pm},\boldsymbol{\lambda}_j)=0,\\
&(-1)^{\alpha_j}[\varphi_j(\mathbf{y}_{m\mp\varepsilon}^{(\alpha_j,j)},t_{m\mp\varepsilon},\boldsymbol{\lambda}_j)-\varphi_j(\mathbf{y}_{m\mp}^{(\alpha_j,j)},t_{m\mp},\boldsymbol{\lambda}_j)]<0,\\
&(-1)^{\beta_j}[\varphi_j(\mathbf{y}_{m\pm\varepsilon}^{(\beta_j,j)},t_{m\pm\varepsilon},\boldsymbol{\lambda}_j)-\varphi_j(\mathbf{y}_{m\pm}^{(\beta_j,j)},t_{m\pm},\boldsymbol{\lambda}_j)]<0.
\end{aligned}\right\}\tag{4.21}$$

4.3 Synchronicity with Singularity to Constraints

As discussed in Chap. 3, the synchronicity of two dynamical systems to multiple constraints with higher-order singularity can be presented through the following definitions.

Definition 4.12 Consider two dynamical systems in Eqs. (3.1) and (3.2) with constraints in Eq. (3.4). For $\mathbf{y}_{m\pm}^{(\alpha_j,j)}\in\Omega_{(\alpha_j,j)}$ ($\alpha_j\in\mathscr{I}$ and $j\in\mathscr{L}$ with $\mathscr{I}=\{1,2\}$ and $\mathscr{L}=\{1,2,\cdots,l\}$) and $\mathbf{y}_m^{(0,j)}\in\partial\Omega_{(12,j)}$ at time t_m, $\mathbf{y}_{m\pm}^{(\alpha_j,j)}=\mathbf{y}_m^{(0,j)}$. For any small $\varepsilon>0$, there is a time interval $[t_{m-\varepsilon},t_m)$ or $(t_m,t_{m+\varepsilon}]$. At $\mathbf{y}^{(\alpha_j,j)}\in\Omega_{(\alpha_j,j)}^{\pm\varepsilon}$ for time $t\in[t_{m-\varepsilon},t_m)$ or $(t_m,t_{m+\varepsilon}]$, the constraint function $\varphi_j(\mathbf{y}^{(\alpha_j,j)},t,\boldsymbol{\lambda}_j)$ is $C^{r_{\alpha_j}}$ continuous ($r_{\alpha_j}\geq 2k_{\alpha_j}+1$) and $|\varphi_j^{(r_{\alpha_j}+1)}(\mathbf{y}^{(\alpha_j,j)},t,\boldsymbol{\lambda}_j)|<\infty$. The two dynamical systems in Eqs. (3.1) and (3.2) with constraints in Eq. (3.4) are called an l_1-dimensional synchronization with the $(2\mathbf{k}_{\alpha 1}:2\mathbf{k}_{\beta 1})$-order singularity, l_2-dimensional desynchronization with the

$(2\mathbf{k}_{\alpha 2} : 2\mathbf{k}_{\beta 2})$-order singularity, and l_3-dimensional penetration with the $(2\mathbf{k}_{\alpha 3} : 2\mathbf{k}_{\beta 3})$-order singularity for time $t_m \in [t_{m_1}, t_{m_2}]$

(i) if for $\alpha_{j_1} = 1,2$ and $j_1 \in \mathscr{L}_1$

$$\left.\begin{aligned}
&\varphi_{j_1}(\mathbf{y}_{m-}^{(\alpha_{j_1},j_1)}, t_{m-}, \boldsymbol{\lambda}_{j_1}) = \varphi_{j_1}(\mathbf{y}_m^{(0,j_1)}, t_m, \boldsymbol{\lambda}_{j_1}) = 0,\\
&\varphi_{j_1}^{(s_{\alpha_{j_1}})}(\mathbf{y}_{m-}^{(\alpha_{j_1},j_1)}, t_{m-}, \boldsymbol{\lambda}_{j_1}) = 0,\ \text{for } s_{\alpha_{j_1}} = 1, 2, \cdots, 2k_{\alpha_{j_1}},\\
&(-1)^{\alpha_{j_1}}[\varphi_{j_1}(\mathbf{y}_{m-\varepsilon}^{(\alpha_{j_1},j_1)}, t_{m-\varepsilon}, \boldsymbol{\lambda}_{j_1}) - \varphi_{j_1}(\mathbf{y}_{m-}^{(\alpha_{j_1},j_1)}, t_{m-}, \boldsymbol{\lambda}_{j_1})] < 0,
\end{aligned}\right\} \tag{4.22}$$

(ii) if for $\alpha_{j_2} = 1,2$ and $j_2 \in \mathscr{L}_2$

$$\left.\begin{aligned}
&\varphi_{j_2}(\mathbf{y}_{m+}^{(\alpha_{j_2},j_2)}, t_{m+}, \boldsymbol{\lambda}_{j_2}) = \varphi_{j_2}(\mathbf{y}_m^{(0,j_2)}, t_m, \boldsymbol{\lambda}_{j_2}) = 0,\\
&\varphi_{j_2}^{(s_{\alpha_{j_2}})}(\mathbf{y}_{m+}^{(\alpha_{j_2},j_2)}, t_{m+}, \boldsymbol{\lambda}_{j_2}) = 0 \ \text{for } s_{\alpha_{j_2}} = 1, 2, \cdots, 2k_{\alpha_{j_2}},\\
&(-1)^{\alpha_{j_2}}[\varphi_{j_2}(\mathbf{y}_{m+\varepsilon}^{(\alpha_{j_2},j_2)}, t_{m+\varepsilon}, \boldsymbol{\lambda}_{j_2}) - \varphi_{j_2}(\mathbf{y}_{m+}^{(\alpha_{j_2},j_2)}, t_{m+}, \boldsymbol{\lambda}_{j_2})] < 0,
\end{aligned}\right\} \tag{4.23}$$

(iii) if for $\alpha_{j_3}, \beta_{j_3} \in \mathscr{I}$ and $j_3 \in \mathscr{L}_3$ with $\alpha_{j_3} \neq \beta_{j_3}$

$$\begin{aligned}
&\varphi_{j_3}(\mathbf{y}_{m-}^{(\alpha_{j_3},j_3)}, t_{m-}, \boldsymbol{\lambda}_{j_3}) = \varphi_{j_3}(\mathbf{y}_m^{(0,j_3)}, t_m, \boldsymbol{\lambda}_{j_3}) = 0,\\
&\varphi_{j_3}^{(s_{\alpha_{j_3}})}(\mathbf{y}_{m-}^{(\alpha_{j_3},j_3)}, t_{m-}, \boldsymbol{\lambda}_{j_3}) = 0 \ (s_{\alpha_{j_3}} = 0, 1, 2, \cdots, 2k_{\alpha_{j_3}}),\\
&(-1)^{\alpha_{j_3}}[\varphi_{j_3}(\mathbf{y}_{m-\varepsilon}^{(\alpha_{j_3},j_3)}, t_{m-\varepsilon}, \boldsymbol{\lambda}_{j_3}) - \varphi_{j_3}(\mathbf{y}_{m-}^{(\alpha_{j_3},j_3)}, t_{m-}, \boldsymbol{\lambda}_{j_3})] < 0,
\end{aligned} \tag{4.24}$$

$$\left.\begin{aligned}
&\varphi_{j_3}(\mathbf{y}_{m+}^{(\beta_{j_3},j_3)}, t_{m+}, \boldsymbol{\lambda}_{j_3}) = \varphi_{j_3}(\mathbf{y}_m^{(0,j_3)}, t_m, \boldsymbol{\lambda}_{j_3}) = 0,\\
&\varphi_{j_3}^{(s_{\beta_{j_3}})}(\mathbf{y}_{m+}^{(\beta_{j_3},j_3)}, t_{m+}, \boldsymbol{\lambda}_{j_3}) = 0,\ \text{for } s_{\beta_{j_3}} = 1, 2, \cdots, 2k_{\beta_{j_3}},\\
&(-1)^{\beta_{j_3}}[\varphi_{j_3}(\mathbf{y}_{m+\varepsilon}^{(\beta_{j_3},j_3)}, t_{m+\varepsilon}, \boldsymbol{\lambda}_{j_3}) - \varphi_{j_3}(\mathbf{y}_{m+}^{(\beta_{j_3},j_3)}, t_{m+}, \boldsymbol{\lambda}_{j_3})] < 0.
\end{aligned}\right\} \tag{4.25}$$

Notice that $2\mathbf{k}_{\alpha i} = (2k_{\alpha_1}, 2k_{\alpha_2}, \cdots, 2k_{\alpha_{j_i}}, \cdots 2k_{\alpha_{li}})^{\mathrm{T}}$ ($i = 1, 2, 3$). Consider two dynamical systems with l-constraints to be synchronized, or desynchronized or penetrated with higher-order singularity. The corresponding descriptions for such synchronicity will be given as follows.

Definition 4.13 Consider two dynamical systems in Eqs. (3.1) and (3.2) with constraints in Eq. (3.4). For $\mathbf{y}_{m\pm}^{(\alpha_j,j)} \in \boldsymbol{\Omega}_{(\alpha_j,j)}$ ($\alpha_j \in \mathscr{I}$ and $j \in \mathscr{L}$ with $\mathscr{I} = \{1, 2\}$ and $\mathscr{L} = \{1, 2, \cdots, l\}$) and $\mathbf{y}_m^{(0,j)} \in \partial\boldsymbol{\Omega}_{(12,j)}$ at time t_m, $\mathbf{y}_{m\pm}^{(\alpha_j,j)} = \mathbf{y}_m^{(0,j)}$. For any small $\varepsilon > 0$, there is a time interval $[t_{m-\varepsilon}, t_m)$ or $(t_m, t_{m+\varepsilon}]$. At $\mathbf{y}^{(\alpha_j,j)} \in \boldsymbol{\Omega}_{(\alpha_j,j)}^{\pm\varepsilon}$ for time $t \in [t_{m-\varepsilon}, t_m)$ or $(t_m, t_{m+\varepsilon}]$, the constraint function $\varphi_j(\mathbf{y}^{(\alpha_j,j)}, t, \boldsymbol{\lambda}_j)$ is $C^{r_{\alpha_j}}$-continuous and $|\varphi_j^{(r_{\alpha_j}+1)}(\mathbf{y}^{(\alpha_j,j)}, t, \boldsymbol{\lambda}_j)| < \infty$ ($r_{\alpha_j} \geq 2k_{\alpha_j} + 1$). The two dynamical systems in Eqs. (3.1) and (3.2) with constraints in Eq. (3.4) are called an l-dimensional synchronization with the $(2\mathbf{k}_\alpha : 2\mathbf{k}_\beta)$-order singularity for time $t_m \in [t_{m_1}, t_{m_2}]$ if for $\alpha_j = 1{,}2$ and $j \in \mathscr{L}$

$$\left.\begin{aligned} &\varphi_j(\mathbf{y}_{m-}^{(\alpha_j,j)}, t_{m-}, \boldsymbol{\lambda}_j) = \varphi_j(\mathbf{y}_m^{(0,j)}, t_m, \boldsymbol{\lambda}_j) = 0, \\ &\varphi_j^{(s_{\alpha_j})}(\mathbf{y}_{m-}^{(\alpha_j,j)}, t_{m-}, \boldsymbol{\lambda}_j) = 0, \text{ for } s_{\alpha_j} = 1, 2, \cdots, 2k_{\alpha_j}, \\ &(-1)^{\alpha_j}[\varphi_j(\mathbf{y}_{m-\varepsilon}^{(\alpha_j,j)}, t_{m-\varepsilon}, \boldsymbol{\lambda}_j) - \varphi_j(\mathbf{y}_{m-}^{(\alpha_j,j)}, t_{m-}, \boldsymbol{\lambda}_j)] < 0. \end{aligned}\right\} \tag{4.26}$$

Definition 4.14 Consider two dynamical systems in Eqs. (3.1) and (3.2) with constraints in Eq. (3.4). For $\mathbf{y}_{m\pm}^{(\alpha_j,j)} \in \Omega_{(\alpha_j,j)}$ ($\alpha_j \in \mathscr{I}$ and $j \in \mathscr{L}$ with $\mathscr{I} = \{1,2\}$ and $\mathscr{L} = \{1, 2, \cdots, l\}$) and $\mathbf{y}_m^{(0,j)} \in \partial\Omega_{(12,j)}$ at time t_m, $\mathbf{y}_{m\pm}^{(\alpha_j,j)} = \mathbf{y}_m^{(0,j)}$. For any small $\varepsilon > 0$, there is a time interval $[t_{m-\varepsilon}, t_m)$ or $(t_m, t_{m+\varepsilon}]$. At $\mathbf{y}^{(\alpha_j,j)} \in \Omega_{(\alpha_j,j)}^{\pm\varepsilon}$ for time $t \in [t_{m-\varepsilon}, t_m)$ or $(t_m, t_{m+\varepsilon}]$, the constraint function $\varphi_j(\mathbf{y}^{(\alpha_j,j)}, t, \boldsymbol{\lambda}_j)$ is $C^{r_{\alpha_j}}$ continuous and $|\varphi_j^{(r_{\alpha_j}+1)}(\mathbf{y}^{(\alpha_j,j)}, t, \boldsymbol{\lambda}_j)| < \infty$ ($r_{\alpha_j} \geq 2k_{\alpha_j} + 1$). The two dynamical systems in Eqs. (3.1) and (3.2) with constraints in Eq. (3.4) are called an l-dimensional desynchronization with the $(2\mathbf{k}_\alpha : 2\mathbf{k}_\beta)$-order singularity for time $t_m \in [t_{m_1}, t_{m_2}]$ if for $\alpha_j = 1{,}2$ and $j \in \mathscr{L}$

$$\left.\begin{aligned} &\varphi_j(\mathbf{y}_{m+}^{(\alpha_j,j)}, t_{m+}, \boldsymbol{\lambda}_j) = \varphi_j(\mathbf{y}_m^{(0,j)}, t_m, \boldsymbol{\lambda}_j) = 0, \\ &\varphi_j^{(s_{\alpha_j})}(\mathbf{y}_{m+}^{(\alpha_j,j)}, t_{m+}, \boldsymbol{\lambda}_j) = 0 \text{ for } s_{\alpha_j} = 1, 2, \cdots, 2k_{\alpha_j}, \\ &(-1)^{\alpha_j}[\varphi_j(\mathbf{y}_{m+\varepsilon}^{(\alpha_j,j)}, t_{m+\varepsilon}, \boldsymbol{\lambda}_j) - \varphi_j(\mathbf{y}_{m+}^{(\alpha_j,j)}, t_{m+}, \boldsymbol{\lambda}_j)] < 0. \end{aligned}\right\} \tag{4.27}$$

Definition 4.15 Consider two dynamical systems in Eqs. (3.1) and (3.2) with constraints in Eq. (3.4). For $\mathbf{y}_{m\pm}^{(\alpha_j,j)} \in \Omega_{(\alpha_j,j)}$ ($\alpha_j \in \mathscr{I}$ and $j \in \mathscr{L}$ with $\mathscr{I} = \{1,2\}$ and $\mathscr{L} = \{1, 2, \cdots, l\}$) and $\mathbf{y}_m^{(0,j)} \in \partial\Omega_{(12,j)}$ at time t_m, $\mathbf{y}_{m\pm}^{(\alpha_j,j)} = \mathbf{y}_m^{(0,j)}$. For any small $\varepsilon > 0$, there is a time interval $[t_{m-\varepsilon}, t_m)$ or $(t_m, t_{m+\varepsilon}]$. At $\mathbf{y}^{(\alpha_j,j)} \in \Omega_{(\alpha_j,j)}^{\pm\varepsilon}$ for time $t \in [t_{m-\varepsilon}, t_m)$ or $(t_m, t_{m+\varepsilon}]$, the constraint function $\varphi_j(\mathbf{y}^{(\alpha_j,j)}, t, \boldsymbol{\lambda}_j)$ is $C^{r_{\alpha_j}}$ continuous and $|\varphi_j^{(r_{\alpha_j}+1)}(\mathbf{y}^{(\alpha_j,j)}, t, \boldsymbol{\lambda}_j)| < \infty$ ($r_{\alpha_j} \geq 2k_{\alpha_j} + 1$). The two dynamical systems in Eqs. (3.1) and (3.2) with constraints in Eq. (3.4) are called an l-dimensional penetration with the $(2\mathbf{k}_\alpha : 2\mathbf{k}_\beta)$-order singularity for time $t_m \in [t_{m_1}, t_{m_2}]$ if for $\alpha_j, \beta_j \in \mathscr{I}$ and $j \in \mathscr{L}$ with $\alpha_j \neq \beta_j$

$$\left.\begin{aligned} &\varphi_j(\mathbf{y}_{m-}^{(\alpha_j,j)}, t_{m-}, \boldsymbol{\lambda}_j) = \varphi_j(\mathbf{y}_m^{(0,j)}, t_m, \boldsymbol{\lambda}_j) = 0, \\ &\varphi_j^{(s_{\alpha_j})}(\mathbf{y}_{m-}^{(\alpha_j,j)}, t_{m-}, \boldsymbol{\lambda}_j) = 0, \text{ for } s_{\alpha_j} = 1, 2, \cdots, 2k_{\alpha_j}, \\ &(-1)^{\alpha_j}[\varphi_j(\mathbf{y}_{m-\varepsilon}^{(\alpha_j,j)}, t_{m-\varepsilon}, \boldsymbol{\lambda}_j) - \varphi_j(\mathbf{y}_{m-}^{(\alpha_j,j)}, t_{m-}, \boldsymbol{\lambda}_j)] < 0, \end{aligned}\right\} \tag{4.28}$$

$$\left.\begin{array}{l}\varphi_j(\mathbf{y}_{m+}^{(\beta_j,j)},t_{m+},\boldsymbol{\lambda}_j)=\varphi_j(\mathbf{y}_m^{(0,j)},t_m,\boldsymbol{\lambda}_j)=0,\\ \varphi_j^{(s_{\beta_j})}(\mathbf{y}_{m+}^{(\beta_j,j)},t_{m+},\boldsymbol{\lambda}_j)=0 \text{ for } s_{\beta_j}=1,2,\cdots,2k_{\beta_j},\\ (-1)^{\beta_j}[\varphi_j(\mathbf{y}_{m+\varepsilon}^{(\beta_j,j)},t_{m+\varepsilon},\boldsymbol{\lambda}_j)-\varphi_j(\mathbf{y}_{m+}^{(\beta_j,j)},t_{m+},\boldsymbol{\lambda}_j)]<0.\end{array}\right\} \tag{4.29}$$

4.4 Higher-Order Singularity to Constraints

Since a resultant flow of two dynamical systems to one of l-constraints possesses the higher-order singularity, the synchronicity of the two dynamical systems to the l-constraints will be changed. In this section, the higher-order singularity of a resultant flow of two dynamical systems to the jth constraint boundary from the l-constraints will be presented herein.

Definition 4.16 Consider two dynamical systems in Eqs. (3.1) and (3.2) with constraints in Eq. (3.4). For $\mathbf{y}_{m\pm}^{(\alpha_j,j)}\in\Omega_{(\alpha_j,j)}$ ($\alpha_j\in\mathscr{I}$ and $j\in\mathscr{L}$ with $\mathscr{I}=\{1,2\}$ and $\mathscr{L}=\{1,2,\cdots,l\}$) and $\mathbf{y}_m^{(0,j)}\in\partial\Omega_{(12,j)}$ at time t_m, $\mathbf{y}_{m\pm}^{(\alpha_j,j)}=\mathbf{y}_m^{(0,j)}$. For any small $\varepsilon>0$, there is a time interval $[t_{m-\varepsilon},t_m)$ or $(t_m,t_{m+\varepsilon}]$. At $\mathbf{y}^{(\alpha_j,j)}\in\Omega_{(\alpha_j,j)}^{\pm\varepsilon}$ for time $t\in[t_{m-\varepsilon},t_m)$ or $(t_m,t_{m+\varepsilon}]$, the constraint function $\varphi_j(\mathbf{y}^{(\alpha_j,j)},t,\boldsymbol{\lambda}_j)$ is $C^{r_{\alpha_j}}$ continuous and $|\varphi_j^{(r_{\alpha_j}+1)}(\mathbf{y}^{(\alpha_j,j)},\ t,\boldsymbol{\lambda}_j)|<\infty$ ($r_{\alpha_j}\geq 2k_{\alpha_j}$). A flow of the resultant system of two dynamical systems in Eqs. (3.1) and (3.2) with l-constraints in Eq. (3.4) is called to be *tangential* to the jth constraint boundary with *the* $(2k_{\alpha_j}-1)$th-*order* for time $t_m\in[t_{m_1},t_{m_2}]$ if for $j\in\mathscr{L}$ and $\alpha_j\in\mathscr{I}$

$$\left.\begin{array}{l}\varphi_j(\mathbf{y}_{m\mp}^{(\alpha_j,j)},t_{m\mp},\boldsymbol{\lambda}_j)=\varphi_j(\mathbf{y}_m^{(0,j)},t_m,\boldsymbol{\lambda}_j)=0,\\ \varphi_j^{(1)}(\mathbf{y}_{m\mp}^{(\alpha_j,j)},t_{m\mp},\boldsymbol{\lambda}_j)=0 \text{ for } s_{\alpha_j}=1,2,\cdots,2k_{\alpha_j}-1,\\ (-1)^{\alpha_j}[\varphi_j(\mathbf{y}_{m\mp\varepsilon}^{(\alpha_j,j)},t_{m\mp\varepsilon},\boldsymbol{\lambda}_j)-\varphi_j(\mathbf{y}_{m\mp}^{(\alpha_j,j)},t_{m\mp},\boldsymbol{\lambda}_j)]<0.\end{array}\right\} \tag{4.30}$$

Definition 4.17 Consider two dynamical systems in Eqs. (3.1) and (3.2) with constraints in Eq. (3.4). For $\mathbf{y}_{m\pm}^{(\alpha_j,j)}\in\Omega_{(\alpha_j,j)}$ ($\alpha_j\in\mathscr{I}$ and $j\in\mathscr{L}$ with $\mathscr{I}=\{1,2\}$ and $\mathscr{L}=\{1,2,\cdots,l\}$) and $\mathbf{y}_m^{(0,j)}\in\partial\Omega_{(12,j)}$ at time t_m, $\mathbf{y}_{m\pm}^{(\alpha_j,j)}=\mathbf{y}_m^{(0,j)}$. For any small $\varepsilon>0$, there is a time interval $[t_{m-\varepsilon},t_m)$ or $(t_m,t_{m+\varepsilon}]$. At $\mathbf{y}^{(\alpha_j,j)}\in\Omega_{(\alpha_j,j)}^{\pm\varepsilon}$ for time $t\in[t_{m-\varepsilon},t_m)$ or $(t_m,t_{m+\varepsilon}]$, the constraint function $\varphi_j(\mathbf{y}^{(\alpha_j,j)},t,\boldsymbol{\lambda}_j)$ is $C^{r_{\alpha_j}}$ continuous and $|\varphi_j^{(r_{\alpha_j}+1)}(\mathbf{y}^{(\alpha_j,j)},t,\boldsymbol{\lambda}_j)|<\infty$ ($r_{\alpha_j}\geq 2k_{\alpha_j}+1$).

(i) The synchronization of the $(2k_{\alpha_j}:2k_{\beta_j})$-order of the two dynamical systems in Eqs. (3.1) and (3.2) with the jth constraint in Eq. (3.4) is said to *be vanishing* to form the penetration from domain $\Omega_{(\alpha_j j)}$ to $\Omega_{(\beta_j j)}$ on the jth constraint boundary at time t_m if for $\alpha_j,\beta_j\in\mathscr{I}$ and $\beta_j\neq\alpha_j$ with $j\in\mathscr{L}$

$$\left.\begin{array}{l}\varphi_j(\mathbf{y}_{m-}^{(\alpha_j,j)},t_{m-},\boldsymbol{\lambda}_j)=\varphi_j(\mathbf{y}_{m\mp}^{(\beta_j,j)},t_{m\mp},\boldsymbol{\lambda}_j)=\varphi_j(\mathbf{y}_m^{(0,j)},t_m,\boldsymbol{\lambda}_j)=0,\\ \varphi_j^{(s_{\alpha_j})}(\mathbf{y}_{m-}^{(\alpha_j,j)},t_{m-},\boldsymbol{\lambda}_j)=0 \text{ for } s_{\alpha_j}=1,2,\cdots,2k_{\alpha_j},\\ \varphi_j^{(s_{\beta_j})}(\mathbf{y}_{m\mp}^{(\beta_j,j)},t_{m\mp},\boldsymbol{\lambda}_j)=0 \text{ for } s_{\beta_j}=1,2,\cdots,2k_{\beta_j}+1,\\ (-1)^{\alpha_j}[\varphi_j(\mathbf{y}_{m-\varepsilon}^{(\alpha_j,j)},t_{m-\varepsilon},\boldsymbol{\lambda}_j)-\varphi_j(\mathbf{y}_{m-}^{(\alpha_j,j)},t_{m-},\boldsymbol{\lambda}_j)]<0,\\ (-1)^{\beta_j}[\varphi_j(\mathbf{y}_{m\mp\varepsilon}^{(\beta_j,j)},t_{m\mp\varepsilon},\boldsymbol{\lambda}_j)-\varphi_j(\mathbf{y}_{m\mp}^{(\beta_j,j)},t_{m\mp},\boldsymbol{\lambda}_j)]<0.\end{array}\right\} \quad (4.31)$$

(ii) The synchronization of the $(2k_{\alpha_j}:2k_{\beta_j})$-order of the two dynamical systems in Eqs. (3.1) and (3.2) with the jth constraint in Eq. (3.4) is said to be onset from the penetration on the jth constraint boundary from domain $\Omega_{(\alpha_j j)}$ to $\Omega_{(\beta_j j)}$ at time t_m if for $\alpha_j,\beta_j\in\mathscr{I}$ and $\beta_j\neq\alpha_j$ with $j\in\mathscr{L}$

$$\left.\begin{array}{l}\varphi_j(\mathbf{y}_{m-}^{(\alpha_j,j)},t_{m-},\boldsymbol{\lambda}_j)=\varphi_j(\mathbf{y}_{m\pm}^{(\beta_j,j)},t_{m\pm},\boldsymbol{\lambda}_j)=\varphi_j(\mathbf{y}_m^{(0,j)},t_m,\boldsymbol{\lambda}_j)=0,\\ \varphi_j^{(s_{\alpha_j})}(\mathbf{y}_{m-}^{(\alpha_j,j)},t_{m-},\boldsymbol{\lambda}_j)=0 \text{ for } s_{\alpha_j}=1,2,\cdots,2k_{\alpha_j},\\ \varphi_j^{(s_{\beta_j})}(\mathbf{y}_{m\pm}^{(\beta_j,j)},t_{m\pm},\boldsymbol{\lambda}_j)=0 \text{ for } s_{\beta_j}=1,2,\cdots,2k_{\beta_j}+1,\\ (-1)^{\alpha_j}[\varphi_j(\mathbf{y}_{m-\varepsilon}^{(\alpha_j,j)},t_{m-\varepsilon},\boldsymbol{\lambda}_j)-\varphi_j(\mathbf{y}_{m-}^{(\alpha_j,j)},t_{m-},\boldsymbol{\lambda}_j)]<0,\\ (-1)^{\beta_j}[\varphi_j(\mathbf{y}_{m\pm\varepsilon}^{(\beta_j,j)},t_{m\pm\varepsilon},\boldsymbol{\lambda}_j)-\varphi_j(\mathbf{y}_{m\pm}^{(\beta_j,j)},t_{m\pm},\boldsymbol{\lambda}_j)]<0.\end{array}\right\} \quad (4.32)$$

Definition 4.18 Consider two dynamical systems in Eqs. (3.1) and (3.2) with constraints in Eq. (3.4). For $\mathbf{y}_{m\pm}^{(\alpha_j,j)}\in\Omega_{(\alpha_j,j)}$ ($\alpha_j\in\mathscr{I}$ and $j\in\mathscr{L}$ with $\mathscr{I}=\{1,2\}$ and $\mathscr{L}=\{1,2,\cdots,l\}$) and $\mathbf{y}_m^{(0,j)}\in\partial\Omega_{(12,j)}$ at time t_m, $\mathbf{y}_{m\pm}^{(\alpha_j,j)}=\mathbf{y}_m^{(0,j)}$. For any small $\varepsilon>0$, there is a time interval $[t_{m-\varepsilon},t_m)$ or $(t_m,t_{m+\varepsilon}]$. At $\mathbf{y}^{(\alpha_j,j)}\in\Omega_{(\alpha_j,j)}^{\pm\varepsilon}$ for time $t\in[t_{m-\varepsilon},t_m)$ or $(t_m,t_{m+\varepsilon}]$, the constraint function $\varphi_j(\mathbf{y}^{(\alpha_j,j)},t,\boldsymbol{\lambda}_j)$ is $C^{r_{\alpha_j}}$ continuous and $|\varphi_j^{(r_{\alpha_j}+1)}(\mathbf{y}^{(\alpha_j,j)},t,\boldsymbol{\lambda}_j)|<\infty$ $(r_{\alpha_j}\geq 2k_{\alpha_j}+1)$.

(i) The $(2k_{\alpha_j}:2k_{\beta_j})$-synchronization of the two dynamical systems in Eqs. (3.1) and (3.2) with the jth constraint in Eq. (3.4) is called to be onset from the desynchronization on the jth constraint boundary at time t_m if for $\alpha_j,\beta_j\in\mathscr{I}$ and $\beta_j\neq\alpha_j$ with $j\in\mathscr{L}$

$$\left.\begin{array}{l}\varphi_j(\mathbf{y}_{m\pm}^{(\alpha_j,j)},t_{m\pm},\boldsymbol{\lambda}_j)=\varphi_j(\mathbf{y}_{m\pm}^{(\beta_j,j)},t_{m\pm},\boldsymbol{\lambda}_j)=\varphi_j(\mathbf{y}_m^{(0,j)},t_m,\boldsymbol{\lambda}_j)=0,\\ \varphi_j^{(s_{\alpha_j})}(\mathbf{y}_{m\pm}^{(\alpha_j,j)},t_{m\pm},\boldsymbol{\lambda}_j)=0 \text{ for } s_{\alpha_j}=1,2,\cdots,2k_{\alpha_j}+1,\\ \varphi_j^{(s_{\beta_j})}(\mathbf{y}_{m\pm}^{(\beta_j,j)},t_{m\pm},\boldsymbol{\lambda}_j)=0 \text{ for } s_{\beta_j}=1,2,\cdots,2k_{\beta_j}+1,\\ (-1)^{\alpha_j}[\varphi_j(\mathbf{y}_{m\pm\varepsilon}^{(\alpha_j,j)},t_{m\pm\varepsilon},\boldsymbol{\lambda}_j)-\varphi_j(\mathbf{y}_{m\pm}^{(\alpha_j,j)},t_{m\pm},\boldsymbol{\lambda}_j)]<0,\\ (-1)^{\beta_j}[\varphi_j(\mathbf{y}_{m\pm\varepsilon}^{(\beta_j,j)},t_{m\pm\varepsilon},\boldsymbol{\lambda}_j)-\varphi_j(\mathbf{y}_{m\pm}^{(\beta_j,j)},t_{m\pm},\boldsymbol{\lambda}_j)]<0.\end{array}\right\} \quad (4.33)$$

(ii) The $(2k_{\alpha_j}:2k_{\beta_j})$-synchronization of the two dynamical systems in Eqs. (3.1) and (3.2) with the jth constraint in Eq. (3.4) is said to be *vanishing* to form the desynchronization on the jth constraint boundary at time t_m if

$$\left.\begin{array}{l}\varphi_j(\mathbf{y}_{m\mp}^{(\alpha_j,j)},t_{m\mp},\boldsymbol{\lambda}_j)=\varphi_j(\mathbf{y}_{m\mp}^{(\beta_j,j)},t_{m\mp},\boldsymbol{\lambda}_j)=\varphi_j(\mathbf{y}_m^{(0,j)},t_m,\boldsymbol{\lambda}_j)=0,\\ \varphi_j^{(s_{\alpha_j})}(\mathbf{y}_{m\mp}^{(\alpha_j,j)},t_{m\mp},\boldsymbol{\lambda}_j)=0 \text{ for } s_{\alpha_j}=1,2,\cdots,2k_{\alpha_j}+1,\\ \varphi_j^{(s_{\beta_j})}(\mathbf{y}_{m\mp}^{(\beta_j,j)},t_{m\mp},\boldsymbol{\lambda}_j)=0 \text{ for } s_{\beta_j}=1,2,\cdots,2k_{\beta_j}+1,\\ (-1)^{\alpha_j}[\varphi_j(\mathbf{y}_{m\mp\varepsilon}^{(\alpha_j,j)},t_{m\mp\varepsilon},\boldsymbol{\lambda}_j)-\varphi_j(\mathbf{y}_{m\mp}^{(\alpha_j,j)},t_{m\mp},\boldsymbol{\lambda}_j)]<0,\\ (-1)^{\beta_j}[\varphi_j(\mathbf{y}_{m\mp\varepsilon}^{(\beta_j,j)},t_{m\mp\varepsilon},\boldsymbol{\lambda}_j)-\varphi_j(\mathbf{y}_{m\mp}^{(\beta_j,j)},t_{m\mp},\boldsymbol{\lambda}_j)]<0.\end{array}\right\} \quad (4.34)$$

Definition 4.19 Consider two dynamical systems in Eqs. (3.1) and (3.2) with constraints in Eq. (3.4). For $\mathbf{y}_{m\pm}^{(\alpha_j,j)}\in\Omega_{(\alpha_j,j)}$ ($\alpha_j\in\mathscr{I}$ and $j\in\mathscr{L}$ with $\mathscr{I}=\{1,2\}$ and $\mathscr{L}=\{1,2,\cdots,l\}$) and $\mathbf{y}_m^{(0,j)}\in\partial\Omega_{(12,j)}$ at time t_m, $\mathbf{y}_{m\pm}^{(\alpha_j,j)}=\mathbf{y}_m^{(0,j)}$. For any small $\varepsilon>0$, there is a time interval $[t_{m-\varepsilon},t_m)$ or $(t_m,t_{m+\varepsilon}]$. At $\mathbf{y}^{(\alpha_j,j)}\in\Omega_{(\alpha_j,j)}^{\pm\varepsilon}$ for time $t\in[t_{m-\varepsilon},t_m)$ or $(t_m,t_{m+\varepsilon}]$, the constraint function $\varphi_j(\mathbf{y}^{(\alpha_j,j)},t,\boldsymbol{\lambda}_j)$ is $C^{r_{\alpha_j}}$ continuous and $|\varphi_j^{(r_{\alpha_j}+1)}(\mathbf{y}^{(\alpha_j,j)},t,\boldsymbol{\lambda}_j)|<\infty$ ($r_{\alpha_j}\geq 2k_{\alpha_j}+1$).

(i) The $(2k_{\alpha_j}:2k_{\beta_j})$-desynchronization of the two dynamical systems in Eqs. (3.1) and (3.2) with the jth constraint in Eq. (3.4) is called *to be vanishing* to form the penetration on the jth constraint boundary from domain $\Omega_{(\alpha_j,j)}$ to $\Omega_{(\beta_j,j)}$ at time t_m if for $\alpha_j,\beta_j\in\mathscr{I}$ and $\beta_j\neq\alpha_j$ with $j\in\mathscr{L}$

$$\left.\begin{array}{l}\varphi_j(\mathbf{y}_{m\pm}^{(\alpha_j,j)},t_{m\pm},\boldsymbol{\lambda}_j)=\varphi_j(\mathbf{y}_{m+}^{(\beta_j,j)},t_{m+},\boldsymbol{\lambda}_j)=\varphi_j(\mathbf{y}_m^{(0,j)},t_m,\boldsymbol{\lambda}_j)=0;\\ \varphi_j^{(s_{\alpha_j})}(\mathbf{y}_{m\pm}^{(\alpha_j,j)},t_{m\pm},\boldsymbol{\lambda}_j)=0 \text{ for } s_{\alpha_j}=1,2,\cdots,2k_{\alpha_j}+1;\\ \varphi_j^{(s_{\beta_j})}(\mathbf{y}_{m+}^{(\beta_j,j)},t_{m+},\boldsymbol{\lambda}_j)=0 \text{ for } s_{\beta_j}=1,2,\cdots,2k_{\beta_j};\\ (-1)^{\alpha_j}[\varphi_j(\mathbf{y}_{m\pm\varepsilon}^{(\alpha_j,j)},t_{m\pm\varepsilon},\boldsymbol{\lambda}_j)-\varphi_j(\mathbf{y}_{m\pm}^{(\alpha_j,j)},t_{m\pm},\boldsymbol{\lambda}_j)]<0,\\ (-1)^{\beta_j}[\varphi_j(\mathbf{y}_{m+\varepsilon}^{(\beta_j,j)},t_{m+\varepsilon},\boldsymbol{\lambda}_j)-\varphi_j(\mathbf{y}_{m+}^{(\beta_j,j)},t_{m+},\boldsymbol{\lambda}_j)]<0.\end{array}\right\} \quad (4.35)$$

(ii) The $(2k_{\alpha_j}:2k_{\beta_j})$-synchronization of the two dynamical systems in Eqs. (3.1) and (3.2) with the jth constraint in Eq. (3.4) is said to be onset from the penetration on the jth constraint boundary from domain $\Omega_{(\alpha_j,j)}$ to $\Omega_{(\beta_j,j)}$ at time t_m if for $\alpha_j,\beta_j\in\mathscr{I}$ and $\beta_j\neq\alpha_j$ with $j\in\mathscr{L}$

$$\begin{array}{l}\varphi_j(\mathbf{y}_{m\mp}^{(\alpha_j,j)},t_{m\mp},\boldsymbol{\lambda}_j)=\varphi_j(\mathbf{y}_{m+}^{(\beta_j,j)},t_{m+},\boldsymbol{\lambda}_j)=\varphi_j(\mathbf{y}_m^{(0,j)},t_m,\boldsymbol{\lambda}_j)=0;\\ \varphi_j^{(s_{\alpha_j})}(\mathbf{y}_{m\mp}^{(\alpha_j,j)},t_{m\mp},\boldsymbol{\lambda}_j)=0 \text{ for } s_{\alpha_j}=1,2,\cdots,2k_{\alpha_j}+1;\end{array} \quad (4.36a)$$

$$\begin{aligned}&\varphi_j^{(s_{\beta_j})}(\mathbf{y}_{m+}^{(\beta_j,j)},t_{m+},\boldsymbol{\lambda}_j)=0 \text{ for } s_{\beta_j}=1,2,\cdots,2k_{\beta_j};\\&(-1)^{\alpha_j}[\varphi_j(\mathbf{y}_{m\mp\varepsilon}^{(\alpha_j,j)},t_{m\mp\varepsilon},\boldsymbol{\lambda}_j)-\varphi_j(\mathbf{y}_{m\mp}^{(\alpha_j,j)},t_{m\mp},\boldsymbol{\lambda}_j)]<0,\\&(-1)^{\beta_j}[\varphi_j(\mathbf{y}_{m+\varepsilon}^{(\beta_j,j)},t_{m+\varepsilon},\boldsymbol{\lambda}_j)-\varphi_j(\mathbf{y}_{m+}^{(\beta_j,j)},t_{m+},\boldsymbol{\lambda}_j)]<0.\end{aligned} \tag{4.36b}$$

Definition 4.20 Consider two dynamical systems in Eqs. (3.1) and (3.2) with constraints in Eq. (3.4). For $\mathbf{y}_{m\pm}^{(\alpha_j,j)}\in\boldsymbol{\Omega}_{(\alpha_j,j)}$ ($\alpha_j\in\mathscr{I}$ and $j\in\mathscr{L}$ with $\mathscr{I}=\{1,2\}$ and $\mathscr{L}=\{1,2,\cdots,l\}$) and $\mathbf{y}_m^{(0,j)}\in\partial\boldsymbol{\Omega}_{(12,j)}$ at time t_m, $\mathbf{y}_{m\pm}^{(\alpha_j,j)}=\mathbf{y}_m^{(0,j)}$. For any small $\varepsilon>0$, there is a time interval $[t_{m-\varepsilon},t_m)$ or $(t_m,t_{m+\varepsilon}]$. At $\mathbf{y}^{(\alpha_j,j)}\in\boldsymbol{\Omega}_{(\alpha_j,j)}^{\pm\varepsilon}$ for time $t\in[t_{m-\varepsilon},t_m)$ or $(t_m,t_{m+\varepsilon}]$, the constraint function $\varphi_j(\mathbf{y}^{(\alpha_j,j)},t,\boldsymbol{\lambda}_j)$ is $C^{r_{\alpha_j}}$-continuous and $|\varphi_j^{(r_{\alpha_j}+1)}(\mathbf{y}^{(\alpha_j,j)},t,\boldsymbol{\lambda}_j)|<\infty$ ($r_{\alpha_j}\geq 2k_{\alpha_j}+1$). The $(2k_{\alpha_j}:2k_{\beta_j})$-penetration of the two dynamical systems in Eqs. (3.1) and (3.2) with the jth constraint in Eq. (3.4) is called to be *switching* from the $(2k_{\beta_j}:2k_{\alpha_j})$-penetration on the jth constraint boundary at time t_m if for $\alpha_j,\beta_j\in\mathscr{I}$ and $\beta_j\neq\alpha_j$ with $j\in\mathscr{L}$

$$\left.\begin{aligned}&\varphi_j(\mathbf{y}_{m\mp}^{(\alpha_j,j)},t_{m\mp},\boldsymbol{\lambda}_j)=\varphi_j(\mathbf{y}_{m\pm}^{(\beta_j,j)},t_{m\pm},\boldsymbol{\lambda}_j)=\varphi_j(\mathbf{y}_m^{(0,j)},t_m,\boldsymbol{\lambda}_j)=0;\\&\varphi_j^{(s_{\alpha_j})}(\mathbf{y}_{m\mp}^{(\alpha_j,j)},t_{m\mp},\boldsymbol{\lambda}_j)=0 \text{ for } s_{\alpha_j}=1,2,\cdots,2k_{\alpha_j}+1;\\&\varphi_j^{(s_{\beta_j})}(\mathbf{y}_{m\pm}^{(\beta_j,j)},t_{m\pm},\boldsymbol{\lambda}_j)=0 \text{ for } s_{\beta_j}=1,2,\cdots,2k_{\beta_j}+1;\\&(-1)^{\alpha_j}[\varphi_j(\mathbf{y}_{m\mp\varepsilon}^{(\alpha_j,j)},t_{m\mp\varepsilon},\boldsymbol{\lambda}_j)-\varphi_j(\mathbf{y}_{m\mp}^{(\alpha_j,j)},t_{m\mp},\boldsymbol{\lambda}_j)]<0,\\&(-1)^{\beta_j}[\varphi_j(\mathbf{y}_{m\pm\varepsilon}^{(\beta_j,j)},t_{m\pm\varepsilon},\boldsymbol{\lambda}_j)-\varphi_j(\mathbf{y}_{m\pm}^{(\beta_j,j)},t_{m\pm},\boldsymbol{\lambda}_j)]<0.\end{aligned}\right\} \tag{4.37}$$

4.5 Synchronization to All Constraints

In this section, the necessary and sufficient conditions for such synchronicity of the two dynamical systems to multi-constraints will be discussed. Because of many constraints for two dynamical systems, the synchronicity for each constraint should be discussed.

Theorem 4.1 *Consider two dynamical systems in Eqs. (3.1) and (3.2) with constraints in Eq. (3.4). For* $\mathbf{y}_{m\pm}^{(\alpha_j,j)}\in\boldsymbol{\Omega}_{(\alpha_j,j)}$ ($\alpha_j\in\mathscr{I}$ and $j\in\mathscr{L}$ *with* $\mathscr{I}=\{1,2\}$ *and* $\mathscr{L}=\{1,2,\cdots,l\}$) *and* $\mathbf{y}_m^{(0,j)}\in\partial\boldsymbol{\Omega}_{(12,j)}$ *at time* t_m, $\mathbf{y}_{m\pm}^{(\alpha_j,j)}=\mathbf{y}_m^{(0,j)}$. *For any small* $\varepsilon>0$, *there is a time interval* $[t_{m-\varepsilon},t_m)$ *or* $(t_m,t_{m+\varepsilon}]$. *The constraint function* $\varphi_j(\mathbf{y}^{(\alpha_j)},t,\boldsymbol{\lambda}_j)$ *is* $C^{r_{\alpha_j}}$*-continuous and* $|\varphi_j^{(r_{\alpha_j}+1)}(\mathbf{y}^{(\alpha_j,j)},t,\boldsymbol{\lambda}_j)|<\infty$ ($r_{\alpha_j}\geq 3$). *For* $\mathbf{y}^{(\alpha_j,j)}\in\boldsymbol{\Omega}_{(\alpha_j,j)}$ *and* $\mathbf{y}^{(0,j)}\in\partial\boldsymbol{\Omega}_{(12,j)}$, *suppose* $D^{s_{\alpha_j}}\mathbb{F}^{(\alpha_j,j)}(\mathbf{y}^{(\alpha_j,j)},t,\boldsymbol{\pi}^{(\alpha_j,j)})=D^{s_{\alpha_j}}\mathbb{F}^{(0,j)}(\mathbf{y}^{(0,j)},t,\boldsymbol{\lambda}_j)$ ($s_{\alpha_j}=0,1,2,\cdots$) *for* $\mathbf{y}^{(\alpha_j,j)}=\mathbf{y}^{(0,j)}$. *The two dynamical systems* *in Eqs. (3.1) and (3.2) to l-constraints in Eq. (3.4) are synchronized with l-dimensions for time* $t \in [t_{m_1}, t_{m_2}]$ *in the sense of Eq. (3.4) if and only if*

(i) *for all* $j \in \mathscr{L}$, $\mathbf{y}_{m\pm}^{(\alpha_j,j)} \in \mathbf{\Omega}_{(\alpha_j,j)}$ *and* $\mathbf{y}_m^{(0,j)} \in \partial\mathbf{\Omega}_{(12,j)}$ *for any time* t_m

$$\begin{aligned}&\mathbf{y}_{m\pm}^{(\alpha_j,j)} = \mathbf{y}_m^{(0,j)}, \varphi_j^{(s_{\alpha_j})}(\mathbf{y}_{m\pm}^{(\alpha_j,j)}, t_m, \boldsymbol{\lambda}_j) = 0\\ &\text{for } \alpha_j = 1,2 \text{ and } s_{\alpha_j} = 0,1,2,\cdots\end{aligned} \tag{4.38}$$

(ii) *for all* $j \in \mathscr{L}$, $\mathbf{y}_{\kappa}^{(\alpha_j,j)} \in \mathbf{\Omega}_{(\alpha_j,j)}^{-\varepsilon}$ *at time* $t_\kappa^- \in [t_{m-\varepsilon}, t_m)$ *and* $\mathbf{y}_m^{(0,j)} \in \partial\mathbf{\Omega}_{(12,j)}$ *with* $t_m \in (t_{m_1}, t_{m_2})$

$$\begin{aligned}&\mathbf{y}_{\kappa}^{(\alpha_j,j)} \neq \mathbf{y}_m^{(0,j)}, (-1)^{\alpha_j}\varphi_j^{(1)}(\mathbf{y}_{\kappa}^{(\alpha_j,j)}, t_\kappa^-, \boldsymbol{\lambda}_j) > 0 \text{ and}\\ &\lim_{t_\kappa^- \to t_m} \varphi_j^{(1)}(\mathbf{y}_{\kappa}^{(\alpha_j,j)}, t_\kappa^-, \boldsymbol{\lambda}_j) = 0 \text{ for } \alpha_j = 1,2\end{aligned} \tag{4.39}$$

(iii) *for the* jth *constraint (* $j \in \mathscr{L}$ *),* $\mathbf{y}_{\kappa}^{(\alpha_j,j)} \in \mathbf{\Omega}_{(\alpha_j,j)}^{+\varepsilon}$ *at time* $t_\kappa^+ \in (t_m, t_{m+\varepsilon}]$ *and* $\mathbf{y}_m^{(0,j)} \in \partial\mathbf{\Omega}_{(12,j)}$ *with* $t_m \notin [t_{m_1}, t_{m_2}]$

$$\begin{aligned}&\mathbf{y}_{\kappa}^{(\alpha_j,j)} \neq \mathbf{y}_m^{(0,j)}, (-1)^{\alpha_j}\varphi_j^{(1)}(\mathbf{y}_{\kappa}^{(\alpha_j,j)}, t_\kappa^+, \boldsymbol{\lambda}_j) < 0 \text{ and}\\ &\lim_{t_\kappa^+ \to t_m} \varphi_j^{(1)}(\mathbf{y}_{\kappa}^{(\alpha_j,j)}, t_\kappa^+, \boldsymbol{\lambda}_j) = 0 \text{ for } \alpha_j = 1,2\end{aligned} \tag{4.40}$$

(iv) *for the* jth *constraint (* $j \in \mathscr{L}$ *),* $\mathbf{y}_{\kappa}^{(\alpha_j,j)} \in \mathbf{\Omega}_{(\alpha_j,j)}^{+\varepsilon}$ *at time* $t_\kappa^- \in [t_{m-\varepsilon}, t_{m-})$, $t_\kappa^+ \in (t_{m+}, t_{m+\varepsilon}]$ *and* $\mathbf{y}_m^{(0,j)} \in \partial\mathbf{\Omega}_{12(j)}$ *with* $t_m = t_{m_1}$ and t_{m_2}

$$\begin{aligned}&\mathbf{y}_{\kappa}^{(\alpha)} \neq \mathbf{y}_m^{(0)}, \lim_{t_\kappa^\pm \to t_{m\pm}} \varphi_j^{(1)}(\mathbf{y}_{\kappa}^{(\alpha_j,j)}, t_\kappa^\pm, \boldsymbol{\lambda}_j) = 0,\\ &\lim_{t_\kappa^\pm \to t_{m\pm}} (-1)^{\alpha_j}\varphi_j^{(2)}(\mathbf{y}_{\kappa}^{(\alpha_j,j)}, t_\kappa^\pm, \boldsymbol{\lambda}_j) < 0 \text{ for } \alpha_j = 1,2\end{aligned} \tag{4.41}$$

Proof The proof is similar to the proof of Theorem 3.1 for each $j \in \mathscr{L}$. If the conditions in Eqs. (4.38) and (4.39) are satisfied, from Definition 4.4, the two dynamical systems in Eqs. (3.1) and (3.2) are synchronized for time $t \in (t_{m_1}, t_{m_2})$ in the sense of Eq. (3.4), vice versa. If the onset and vanishing conditions in Eqs. (4.40) and (4.41) hold, from Definition 4.7, the synchronization of two dynamical systems will start to form and to vanish, vice versa. This theorem is proved. □

Theorem 4.2 *Consider two dynamical systems in Eqs. (3.1) and (3.2) with constraints in Eq. (3.4). For* $\mathbf{y}_{m\pm}^{(\alpha_j,j)} \in \mathbf{\Omega}_{(\alpha_j,j)}$ *(*$\alpha_j \in \mathscr{I}$ and $j \in \mathscr{L}$ *with* $\mathscr{I} = \{1,2\}$ *and* $\mathscr{L} = \{1,2,\cdots,l\}$*) and* $\mathbf{y}_m^{(0,j)} \in \partial\mathbf{\Omega}_{(12,j)}$ *at time* t_m, $\mathbf{y}_{m\pm}^{(\alpha_j,j)} = \mathbf{y}_m^{(0,j)}$. *For any small* $\varepsilon > 0$, *there is a time interval* $[t_{m-\varepsilon}, t_m)$ *or* $(t_m, t_{m+\varepsilon}]$. *The constraint function* $\varphi_j(\mathbf{y}^{(\alpha_j)}, t, \boldsymbol{\lambda}_j)$ *is* $C^{r_{\alpha_j}}$*-continuous and* $|\varphi_j^{(r_{\alpha_j}+1)}(\mathbf{y}^{(\alpha_j,j)}, t, \boldsymbol{\lambda}_j)| < \infty$ *(* $r_{\alpha_j} \geq 2k_{\alpha_j} + 1$ *).*

For $\mathbf{y}^{(\alpha_j,j)} \in \boldsymbol{\Omega}_{(\alpha_j,j)}$ *and* $\mathbf{y}^{(0,j)} \in \partial\boldsymbol{\Omega}_{(12,j)}$, *suppose* $D^{s_{\alpha_j}}\mathbb{F}^{(\alpha_j,j)}(\mathbf{y}^{(\alpha_j,j)}, t, \boldsymbol{\pi}^{(\alpha_j,j)}) = D^{s_{\alpha_j}}\mathbb{F}^{(0,j)}(\mathbf{y}^{(0,j)}, t, \boldsymbol{\lambda}_j)$ *(* $s_{\alpha_j} = 0, 1, 2, \cdots$ *) for* $\mathbf{y}^{(\alpha_j,j)} = \mathbf{y}^{(0,j)}$. *The two dynamical systems in Eqs. (3.1) and (3.2) to l-constraints in Eq. (3.4) are synchronized with l-dimensions for time* $t \in [t_{m_1}, t_{m_2}]$ *in the sense of Eq. (3.4) if and only if*

(i) *for all* $j \in \mathscr{L}$, $\mathbf{y}_{m\pm}^{(\alpha_j,j)} \in \boldsymbol{\Omega}_{(\alpha_j,j)}$ *and* $\mathbf{y}_m^{(0,j)} \in \partial\boldsymbol{\Omega}_{(12,j)}$ *with* $t_m \in (t_{m_1}, t_{m_2})$

$$\begin{aligned} &\mathbf{y}_{m\pm}^{(\alpha_j,j)} = \mathbf{y}_m^{(0,j)}, \varphi_j^{(s_{\alpha_j})}(\mathbf{y}_{m\pm}^{(\alpha_j,j)}, t_m, \boldsymbol{\lambda}_j) = 0 \\ &\text{for } \alpha_j = 1, 2 \text{ and } s_{\alpha_j} = 0, 1, 2, \cdots \end{aligned} \tag{4.42}$$

(ii) *for all* $j \in \mathscr{L}$, $\mathbf{y}_\kappa^{(\alpha_j,j)} \in \boldsymbol{\Omega}_{(\alpha_j,j)}^{-\varepsilon}$ *at time* $t_\kappa^- \in [t_{m-\varepsilon}, t_m)$ *and* $\mathbf{y}_m^{(0,j)} \in \partial\boldsymbol{\Omega}_{(12,j)}$ *with* $t_m \in (t_{m_1}, t_{m_2})$

$$\begin{aligned} &\mathbf{y}_\kappa^{(\alpha_j,j)} \neq \mathbf{y}_m^{(0,j)}, \lim_{t_\kappa^- \to t_{m-}} \varphi_j^{(s_{\alpha_j})}(\mathbf{y}_\kappa^{(\alpha_j,j)}, t_\kappa^-, \boldsymbol{\lambda}_j) = 0 \text{ for } s_{\alpha_j} = 1, 2, \cdots, 2k_{\alpha_j}; \\ &(-1)^{\alpha_j}\varphi_j^{(2k_{\alpha_j}+1)}(\mathbf{y}_\kappa^{(\alpha_j,j)}, t_\kappa^-, \boldsymbol{\lambda}_j) > 0 \text{ and} \\ &\lim_{t_\kappa^- \to t_m} \varphi_j^{(2k_{\alpha_j}+1)}(\mathbf{y}_\kappa^{(\alpha_j,j)}, t_\kappa^-, \boldsymbol{\lambda}_j) = 0 \text{ for } \alpha_j = 1, 2; \end{aligned} \tag{4.43}$$

(iii) *for the* jth *constraint (* $j \in \mathscr{L}$ *),* $\mathbf{y}_\kappa^{(\alpha_j,j)} \in \boldsymbol{\Omega}_{(\alpha_j,j)}^{+\varepsilon}$ *at time* $t_\kappa^+ \in (t_m, t_{m+\varepsilon}]$ *and* $\mathbf{y}_m^{(0,j)} \in \partial\boldsymbol{\Omega}_{(12,j)}$ *with* $t_m \notin [t_{m_1}, t_{m_2}]$

$$\begin{aligned} &\mathbf{y}_\kappa^{(\alpha_j,j)} \neq \mathbf{y}_m^{(0,j)}, \lim_{t_\kappa^- \to t_{m-}} \varphi_j^{(s_{\alpha_j})}(\mathbf{y}_\kappa^{(\alpha_j,j)}, t_\kappa^+, \boldsymbol{\lambda}_j) = 0 \text{ for } s_{\alpha_j} = 1, 2, \cdots, 2k_{\alpha_j}; \\ &(-1)^{\alpha_j}\varphi_j^{(2k_{\alpha_j}+1)}(\mathbf{y}_\kappa^{(\alpha_j,j)}, t_\kappa^+, \boldsymbol{\lambda}_j) < 0 \text{ and} \\ &\lim_{t_\kappa^+ \to t_m} \varphi_j^{(2k_{\alpha_j}+1)}(\mathbf{y}_\kappa^{(\alpha_j,j)}, t_\kappa^+, \boldsymbol{\lambda}_j) = 0 \text{ for } \alpha_j = 1, 2; \end{aligned} \tag{4.44}$$

(iv) *for the* jth *constraint (* $j \in \mathscr{L}$ *),* $\mathbf{y}_\kappa^{(\alpha_j,j)} \in \boldsymbol{\Omega}_{(\alpha_j,j)}^{\pm\varepsilon}$ *at time* $t_\kappa^- \in [t_{m-\varepsilon}, t_{m-})$ *and* $t_\kappa^+ \in (t_{m+}, t_{m+\varepsilon}]$, *and* $\mathbf{y}_m^{(0,j)} \in \partial\boldsymbol{\Omega}_{(12,j)}$ *with* $t_m = t_{m_1}$ and t_{m_2}

$$\begin{aligned} &\mathbf{y}_\kappa^{(\alpha)} \neq \mathbf{y}_m^{(0)}, \lim_{t_\kappa^\pm \to t_{m\pm}} \varphi_j^{(s_{\alpha_j})}(\mathbf{y}_\kappa^{(\alpha_j,j)}, t_\kappa^\pm, \boldsymbol{\lambda}_j) = 0 \text{ for } s_{\alpha_j} = 1, 2, \cdots, 2k_{\alpha_j} + 1; \\ &\lim_{t_\kappa^\pm \to t_{m\pm}} (-1)^{\alpha_j}\varphi_j^{(2k_{\alpha_j}+2)}(\mathbf{y}_\kappa^{(\alpha_j,j)}, t_\kappa^\pm, \boldsymbol{\lambda}_j) < 0 \text{ for } \alpha_j = 1, 2 \end{aligned} \tag{4.45}$$

Proof The proof is similar to the proof of Theorem 3.2 for each $j \in \mathscr{L}$. For all $j \in \mathscr{L}$, if the conditions in Eqs. (4.42) and (4.43) are satisfied, from Definition 4.10, the slave and master systems in Eqs. (3.1) and (3.2) are synchronized for time $t \in (t_{m_1}, t_{m_2})$ in the sense of Eq. (3.4), vice versa. If the onset and vanishing conditions in Eqs. (4.44) and (4.45) hold, from Definition 4.13, the synchronization will start to form and to vanish, vice versa. The proof is completed. □

Theorem 4.3 *Consider two dynamical systems in Eqs. (3.1) and (3.2) with constraints in Eq. (3.4). For* $\mathbf{y}_{m\pm}^{(\alpha_j,j)} \in \boldsymbol{\Omega}_{(\alpha_j,j)}$ *(*$\alpha_j \in \mathscr{I}$ and $j \in \mathscr{L}$ *with* $\mathscr{I} = \{1,2\}$ *and* $\mathscr{L} = \{1,2,\cdots,l\}$*) and* $\mathbf{y}_m^{(0,j)} \in \partial\boldsymbol{\Omega}_{(12,j)}$ *at time* t_m, $\mathbf{y}_{m\pm}^{(\alpha_j,j)} = \mathbf{y}_m^{(0,j)}$. *For any small* $\varepsilon > 0$, *there is a time interval* $[t_{m-\varepsilon}, t_m)$ *or* $(t_m, t_{m+\varepsilon}]$. *The constraint function* $\varphi_j(\mathbf{y}^{(\alpha_j)}, t, \boldsymbol{\lambda}_j)$ *is* $C^{r_{\alpha_j}}$*-continuous and* $|\varphi_j^{(r_{\alpha_j}+1)}(\mathbf{y}^{(\alpha_j,j)}, t, \boldsymbol{\lambda}_j)| < \infty$ *(* $r_{\alpha_j} \geq 3$ *). For* $\mathbf{y}^{(\alpha_j,j)} \in \boldsymbol{\Omega}_{(\alpha_j,j)}$ *and* $\mathbf{y}^{(0,j)} \in \partial\boldsymbol{\Omega}_{12(j)}$, $\mathbb{F}^{(\alpha_j,j)}(\mathbf{y}^{(\alpha_j,j)}, t, \boldsymbol{\pi}^{(\alpha_j,j)}) \neq \mathbb{F}^{(0,j)}(\mathbf{y}^{(0,j)}, t, \boldsymbol{\lambda}_j)$ *is assumed for* $\mathbf{y}^{(\alpha_j,j)} = \mathbf{y}^{(0,j)}$ *and* $\alpha_j \in I$. *The two dynamical systems in Eqs. (3.1) and (3.2) to l-constraints in Eq. (3.4) are synchronized with l-dimensions for time* $t \in [t_{m_1}, t_{m_2}]$ *if and only if*

(i) *for all* $j \in \mathscr{L}$, $\mathbf{y}_{m-}^{(\alpha_j,j)} = \mathbf{y}_m^{(0,j)}$ *and* $\mathbf{y}^{(\alpha_j,j)} \in \boldsymbol{\Omega}_{(\alpha_j,j)}$ *at time* $t_m \in [t_{m_1}, t_{m_2}]$ *(*$\alpha_j \in \mathscr{I}$*)*

$$\varphi_j(\mathbf{y}_{m-}^{(\alpha_j,j)}, t_{m-}, \boldsymbol{\lambda}_j) = \varphi_j(\mathbf{y}_m^{(0,j)}, t_m, \boldsymbol{\lambda}_j) = 0 \tag{4.46}$$

(ii) *for all* $j \in \mathscr{L}$, *time* $t_m \in (t_{m_1}, t_{m_2})$ *and* $\mathbf{y}_{m-}^{(\alpha_j,j)} = \mathbf{y}_m^{(0,j)}$ *with* $\alpha_j = 1,2$

$$(-1)^{\alpha_j}\varphi_j^{(1)}(\mathbf{y}_{m-}^{(\alpha_j,j)}, t_{m-}, \boldsymbol{\lambda}_j) > 0 \tag{4.47}$$

(iii) *with the* jth*-penetration for time* $t = t_{m_i}$, $\mathbf{y}_{m_i}^{(\alpha_j,j)} = \mathbf{y}_{m_i}^{(0)}$ *(*$i = 1,2$*) for* $\alpha_j, \beta_j \in \mathscr{I}$ *and* $\beta_j \neq \alpha_j$

$$\left.\begin{aligned}&\varphi_j^{(1)}(\mathbf{y}_{m_i\pm}^{(\alpha_j,j)}, t_{m_i\pm}, \boldsymbol{\lambda}_j) = 0 \text{ and } (-1)^{\alpha_j}\varphi_j^{(2)}(\mathbf{y}_{m_i\pm}^{(\alpha_j,j)}, t_{m_i\pm}, \boldsymbol{\lambda}_j) < 0,\\&(-1)^{\beta_j}\varphi_j^{(1)}(\mathbf{y}_{m_i-}^{(\beta_j,j)}, t_{m_i-}, \boldsymbol{\lambda}_j) > 0;\end{aligned}\right\} \tag{4.48}$$

or with the jth*-desynchronization for time* $t = t_{m_i}$, $\mathbf{y}_{m_i}^{(\alpha_j,j)} = \mathbf{y}_{m_i}^{(0)}$ *(*$i = 1,2$*) for* $\alpha_j, \beta_j \in \mathscr{I}$ *and* $\beta_j \neq \alpha_j$

$$\left.\begin{aligned}&\varphi_j^{(1)}(\mathbf{y}_{m_i\pm}^{(\alpha_j,j)}, t_{m_i\pm}, \boldsymbol{\lambda}_j) = 0 \text{ and } (-1)^{\alpha_j}\varphi_j^{(2)}(\mathbf{y}_{m_i\pm}^{(\alpha_j,j)}, t_{m_i\pm}, \boldsymbol{\lambda}_j) < 0,\\&\varphi_j^{(1)}(\mathbf{y}_{m_i\pm}^{(\beta_j,j)}, t_{m_i\pm}, \boldsymbol{\lambda}_j) = 0 \text{ and } (-1)^{\beta_j}\varphi_j^{(2)}(\mathbf{y}_{m_i\pm}^{(\beta_j,j)}, t_{m_i\pm}, \boldsymbol{\lambda}_j) > 0.\end{aligned}\right\} \tag{4.49}$$

Proof The proof is similar to the proof of Theorem 3.3 for each $j \in \mathscr{L}$. For all $j \in \mathscr{L}$, if the conditions in Eqs. (4.46) and (4.47) are satisfied, from Definition 4.4, the two dynamical systems in Eqs. (3.1) and (3.2) to l-constraints in Eq. (3.4) are synchronized for time $t \in [t_{m_1}, t_{m_2}]$, vice versa. If the onset and vanishing conditions in Eqs. (4.48) and (4.49) hold, from Definition 4.8 or 4.9, the synchronization of the two dynamical systems to l-constraints in Eq. (3.4) will start to form and to vanish, vice versa. The proof is completed. □

In the foregoing theorem, the synchronization of the slave and master systems is without any singularity except for the onset and vanishing conditions on the boundaries of the constraints. If the synchronization of two dynamical systems with higher-order singularity, the corresponding theorems can be presented as follows.

Theorem 4.4 *Consider two dynamical systems in Eqs. (3.1) and (3.2) with constraints in Eq. (3.4). For* $\mathbf{y}_{m\pm}^{(\alpha_j,j)} \in \boldsymbol{\Omega}_{(\alpha_j,j)}$ ($\alpha_j \in \mathscr{I}$ and $j \in \mathscr{L}$ *with* $\mathscr{I} = \{1,2\}$ *and* $\mathscr{L} = \{1,2,\cdots,l\}$) *and* $\mathbf{y}_m^{(0,j)} \in \partial\boldsymbol{\Omega}_{(12,j)}$ *at time* t_m, $\mathbf{y}_{m\pm}^{(\alpha_j,j)} = \mathbf{y}_m^{(0,j)}$. *For any small* $\varepsilon > 0$, *there is a time interval* $[t_{m-\varepsilon}, t_m)$ *or* $(t_m, t_{m+\varepsilon}]$. *The constraint function* $\varphi_j(\mathbf{y}^{(\alpha_j)}, t, \boldsymbol{\lambda}_j)$ *is* $C^{r_{\alpha_j}}$*-continuous and* $|\varphi_j^{(r_{\alpha_j}+1)}(\mathbf{y}^{(\alpha_j,j)}, t, \boldsymbol{\lambda}_j)| < \infty$ ($r_{\alpha_j} \geq 2k_{\alpha_j} + 1$). *For* $\mathbf{y}^{(\alpha_j,j)} \in \boldsymbol{\Omega}_{(\alpha_j,j)}$ *and* $\mathbf{y}^{(0,j)} \in \partial\boldsymbol{\Omega}_{12(j)}$, $\mathbb{F}^{(\alpha_j,j)}(\mathbf{y}^{(\alpha_j,j)}, t, \boldsymbol{\pi}^{(\alpha_j,j)}) \neq \mathbb{F}^{(0,j)}(\mathbf{y}^{(0,j)}, t, \boldsymbol{\lambda}_j)$ *is assumed for* $\mathbf{y}^{(\alpha_j,j)} = \mathbf{y}^{(0,j)}$ *and* $\alpha_j \in \mathscr{I}$. *The two dynamical systems in Eqs. (3.1) and (3.2) to l-constraints in Eq. (3.4) are synchronized of the* $(2\mathbf{k}_\alpha : 2\mathbf{k}_\beta)$*-type with l-dimensions for time* $t \in [t_{m_1}, t_{m_2}]$ *if and only if*

(i) *for all* $j \in \mathscr{L}$, $\mathbf{y}_{m-}^{(\alpha_j,j)} = \mathbf{y}_m^{(0,j)}$, *and* $\mathbf{y}^{(\alpha_j,j)} \in \boldsymbol{\Omega}_{(\alpha_j,j)}$ *at time* $t_m \in [t_{m_1}, t_{m_2}]$ ($\alpha_j \in \mathscr{I}$)

$$\varphi_j(\mathbf{y}_{m-}^{(\alpha_j,j)}, t_{m-}, \boldsymbol{\lambda}_j) = \varphi_j(\mathbf{y}_m^{(0,j)}, t_m, \boldsymbol{\lambda}_j) = 0 \tag{4.50}$$

(ii) *for all* $j \in \mathscr{L}$, *time* $t_m \in (t_{m_1}, t_{m_2})$ *and* $\mathbf{y}_{m-}^{(\alpha_j,j)} = \mathbf{y}_m^{(0,j)}$ *with* $\alpha_j = 1, 2$

$$\begin{aligned} &(\mathbf{y}_{m-}^{(\alpha_j,j)}, t_{m-}, \boldsymbol{\lambda}_j) = 0 \text{ for } s_{\alpha_j} = 1, 2, \cdots, 2k_{\alpha_j};\\ &(-1)^{\alpha_j}\varphi_j^{(2k_{\alpha_j}+1)}(\mathbf{y}_{m-}^{(\alpha_j,j)}, t_{m-}, \boldsymbol{\lambda}_j) > 0. \end{aligned} \tag{4.51}$$

(iii) *with the* jth *penetration of the* $(2k_{\alpha_j} : 2k_{\beta_j})$*-type at time* $t = t_{m_i}$, $\mathbf{y}_{m-}^{(\alpha_j,j)} = \mathbf{y}_m^{(0,j)}$ ($i = 1, 2$) *for* $\alpha_j, \beta_j \in \mathscr{I}$ and $\beta_j \neq \alpha_j$

$$\left.\begin{aligned} &\varphi_j^{(s_{\alpha_j})}(\mathbf{y}_{m_i\pm}^{(\alpha_j,j)}, t_{m_i\pm}, \boldsymbol{\lambda}_j) = 0 \ (s_{\alpha_j} = 1, 2, \cdots, 2k_{\alpha_j} + 1)\\ &\text{and } (-1)^{\alpha_j}\varphi_j^{(2k_{\alpha_j}+2)}(\mathbf{y}_{m_i\pm}^{(\alpha_j,j)}, t_{m_i\pm}, \boldsymbol{\lambda}_j) < 0;\\ &\varphi_j^{(s_{\beta_j})}(\mathbf{y}_{m_i-}^{(\beta_j,j)}, t_{m_i-}, \boldsymbol{\lambda}_j) = 0 \ (s_{\beta_j} = 1, 2, \cdots, 2k_{\beta_j})\\ &(-1)^{\beta_j}\varphi_j^{(2k_{\beta_j}+1)}(\mathbf{y}_{m_i-}^{(\beta_j,j)}, t_{m_i-}, \boldsymbol{\lambda}_j) > 0; \end{aligned}\right\} \tag{4.52}$$

or with the jth*-desynchronization of the* $(2k_{\alpha_j} : 2k_{\beta_j})$*-type at time* $t = t_{m_i}$, $\mathbf{y}_{m-}^{(\alpha_j,j)} = \mathbf{y}_m^{(0,j)}$ ($i = 1, 2$) *for* $\alpha_j, \beta_j \in \mathscr{I}$ and $\beta_j \neq \alpha_j$

$$\left.\begin{aligned} &\varphi_j^{(s_{\alpha_j})}(\mathbf{y}_{m_i\pm}^{(\alpha_j,j)}, t_{m_i\pm}, \boldsymbol{\lambda}_j) = 0 \ (s_{\alpha_j} = 1, 2, \cdots 2k_{\alpha_j} + 1),\\ &(-1)^{\alpha_j}\varphi_j^{(2)}(\mathbf{y}_{m_i\pm}^{(\alpha_j,j)}, t_{m_i\mp}, \boldsymbol{\lambda}_j) < 0;\\ &\varphi_j^{(s_{\beta_j})}(\mathbf{y}_{m_i\pm}^{(\beta_j,j)}, t_{m_i\pm}, \boldsymbol{\lambda}_j) = 0 \ (s_{\beta_j} = 1, 2, \cdots 2k_{\beta_j} + 1),\\ &(-1)^{\beta_j}\varphi_j^{(2k_{\beta_j}+2)}(\mathbf{y}_{m_i\pm}^{(\beta_j,j)}, t_{m_i\pm}, \boldsymbol{\lambda}_j) > 0. \end{aligned}\right\} \tag{4.53}$$

Proof The proof is similar to the proof of Theorem 3.4 for each $j \in \mathscr{L}$,. For all $j \in \mathscr{L}$, if the conditions in Eqs. (4.50) and (4.51) are satisfied, from Definition 4.8, the slave and master systems in Eqs. (3.1) and (3.2) are synchronized with the $(2\mathbf{k}_\alpha : 2\mathbf{k}_\beta)$-type for time $t \in [t_{m_1}, t_{m_2}]$ in the sense of Eq. (3.4), vice versa. If the onset and vanishing conditions in Eqs. (4.52) and (4.53) hold, from Definition 4.12 or 4.13, the synchronization will start to form and to vanish, vice versa. The proof is completed. □

4.6 Desynchronization to All Constraints

In this section, from the desynchronization of two dynamical systems to multiple constraints, the necessary and sufficient conditions for such desynchronicity will be discussed. Because of many constraints for two dynamical systems, the synchronicity for each single one of constraints should be discussed.

Theorem 4.5 *Consider two dynamical systems in Eqs. (3.1) and (3.2) with constraints in Eq. (3.4). For* $\mathbf{y}_{m\pm}^{(\alpha_j,j)} \in \mathbf{\Omega}_{(\alpha_j,j)}$ *(* $\alpha_j \in \mathscr{I}$ and $j \in \mathscr{L}$ *with* $\mathscr{I} = \{1,2\}$ *and* $\mathscr{L} = \{1,2,\cdots,l\}$*) and* $\mathbf{y}_m^{(0,j)} \in \partial\mathbf{\Omega}_{(12,j)}$ *at time* t_m*,* $\mathbf{y}_{m\pm}^{(\alpha_j,j)} = \mathbf{y}_m^{(0,j)}$*. For any small* $\varepsilon > 0$*, there is a time interval* $[t_{m-\varepsilon}, t_m)$ *or* $(t_m, t_{m+\varepsilon}]$*. The constraint function* $\varphi_j(\mathbf{y}^{(\alpha_j)}, t, \boldsymbol{\lambda}_j)$ *is* $C^{r_{\alpha_j}}$ *continuous and* $|\varphi_j^{(r_{\alpha_j}+1)}(\mathbf{y}^{(\alpha_j,j)}, t, \boldsymbol{\lambda}_j)| < \infty$ *(*$r_{\alpha_j} \geq 3$*). For* $\mathbf{y}^{(\alpha_j,j)} \in \mathbf{\Omega}_{(\alpha_j,j)}$ *and* $\mathbf{y}^{(0,j)} \in \partial\mathbf{\Omega}_{(12,j)}$*, suppose* $D^{s_{\alpha_j}}\mathbb{F}^{(\alpha_j,j)}(\mathbf{y}^{(\alpha_j,j)}, t, \boldsymbol{\pi}^{(\alpha_j,j)}) = D^{s_{\alpha_j}}\mathbb{F}^{(0,j)}(\mathbf{y}^{(0,j)}, t, \boldsymbol{\lambda}_j)$ $(s_{\alpha_j} = 0,\ 1,2,\cdots)$ *for* $\mathbf{y}^{(\alpha_j,j)} = \mathbf{y}^{(0,j)}$*. The two dynamical systems in Eqs. (3.1) and (3.2) to l-constraints in Eq. (3.4) are desynchronized with l-dimensions for time* $t \in [t_{m_1}, t_{m_2}]$ *in the sense of Eq. (3.4) if and only if*

(i) *for all* $j \in \mathscr{L}$, $\mathbf{y}_m^{(\alpha_j,j)} \in \mathbf{\Omega}_{(\alpha_j,j)}$*, and* $\mathbf{y}_m^{(0,j)} \in \partial\mathbf{\Omega}_{(12,j)}$ *for any time* t_m

$$\begin{aligned}&\mathbf{y}_m^{(\alpha_j,j)} = \mathbf{y}_m^{(0,j)}, \varphi_j^{(s_{\alpha_j})}(\mathbf{y}_m^{(\alpha_j,j)}, t_m, \boldsymbol{\lambda}_j) = 0\\&\text{for } \alpha_j = 1,2 \text{ and } s_{\alpha_j} = 0,1,2,\cdots\end{aligned}\tag{4.54}$$

(ii) *for all* $j \in \mathscr{L}$, $\mathbf{y}_\kappa^{(\alpha_j,j)} \in \mathbf{\Omega}_{(\alpha_j,j)}^{+\varepsilon}$ *at time* $t_\kappa^+ \in (t_m, t_{m+\varepsilon}]$ *and* $\mathbf{y}_m^{(0,j)} \in \partial\mathbf{\Omega}_{(12,j)}$ *with* $t_m \in (t_{m_1}, t_{m_2})$

$$\begin{aligned}&\mathbf{y}_\kappa^{(\alpha_j,j)} \neq \mathbf{y}_m^{(0,j)}, (-1)^{\alpha_j}\varphi_j^{(1)}(\mathbf{y}_\kappa^{(\alpha_j,j)}, t_\kappa^+, \boldsymbol{\lambda}_j) < 0 \text{ and}\\&\lim_{t_\kappa^+ \to t_m} \varphi_j^{(1)}(\mathbf{y}_\kappa^{(\alpha_j,j)}, t_\kappa^+, \boldsymbol{\lambda}_j) = 0 \text{ for } \alpha_j = 1,2\end{aligned}\tag{4.55}$$

(iii) *for the* jth *constraint (*$j \in \mathscr{L}$*),* $\mathbf{y}_\kappa^{(\alpha_j,j)} \in \mathbf{\Omega}_{(\alpha_j,j)}^{-\varepsilon}$ *at time* $t_\kappa^- \in [t_{m-\varepsilon}, t_m)$ *and* $\mathbf{y}_m^{(0,j)} \in \partial\mathbf{\Omega}_{(12,j)}$ *with* $t_m \notin [t_{m_1}, t_{m_2}]$.

$$\begin{aligned}&\mathbf{y}_\kappa^{(\alpha_j,j)} \neq \mathbf{y}_m^{(0,j)}, (-1)^{\alpha_j}\varphi_j^{(1)}(\mathbf{y}_\kappa^{(\alpha_j,j)}, t_\kappa^-, \boldsymbol{\lambda}_j) > 0 \text{ and}\\&\lim_{t_\kappa^- \to t_m} \varphi_j^{(1)}(\mathbf{y}_\kappa^{(\alpha_j,j)}, t_\kappa^-, \boldsymbol{\lambda}_j) = 0 \text{ for } \alpha_j = 1,2\end{aligned}\tag{4.56}$$

(iv) *for the* jth *constraint (*$j \in \mathscr{L}$*),* $\mathbf{y}_\kappa^{(\alpha_j,j)} \in \mathbf{\Omega}_{(\alpha_j,j)}^{+\varepsilon}$ *at time* $t_\kappa^- \in [t_{m-\varepsilon}, t_{m-})$*,* $t_\kappa^+ \in (t_{m+}, t_{m+\varepsilon}]$ *and* $\mathbf{y}_m^{(0,j)} \in \partial\mathbf{\Omega}_{(12,j)}$ *with* $t_m = t_{m_1}$ and t_{m_2}

$$\mathbf{y}_{\kappa}^{(\alpha_j,j)} \neq \mathbf{y}_m^{(0,j)}, \lim_{t_{\kappa}^{\pm}\to t_{m\pm}} \varphi_j^{(1)}(\mathbf{y}_{\kappa}^{(\alpha_j,j)}, t_{\kappa}^{\pm}, \boldsymbol{\lambda}_j) = 0 \text{ and}$$

$$\lim_{t_{\kappa}^{\pm}\to t_m} (-1)^{\alpha_j}\varphi_j^{(2)}(\mathbf{y}_{\kappa}^{(\alpha_j,j)}, t_{\kappa}^{\pm}, \boldsymbol{\lambda}_j) < 0 \text{ for } \alpha_j = 1,2. \tag{4.57}$$

Proof The proof is similar to the proof of Theorem 3.1 for each $j \in \mathscr{L}$. For all $j \in \mathscr{L}$, if the conditions in Eqs. (4.54) and (4.55) are satisfied, from Definition 4.5, the two dynamical systems in Eqs. (3.1) and (3.2) to constraints in Eq. (3.4) are desynchronized for time $t \in (t_{m_1}, t_{m_2})$, vice versa. If the onset and vanishing conditions in Eqs. (4.56) and (4.57) hold, from Definition 4.7, the desynchronization will start to form and to vanish, vice versa. This theorem is proved. □

Theorem 4.6 *Consider two dynamical systems in Eqs. (3.1) and (3.2) with constraints in Eq. (3.4). For* $\mathbf{y}_{m\pm}^{(\alpha_j,j)} \in \boldsymbol{\Omega}_{(\alpha_j,j)}$ *(*$\alpha_j \in \mathscr{I}$ *and* $j \in \mathscr{L}$ *with* $\mathscr{I} = \{1,2\}$ *and* $\mathscr{L} = \{1,2,\cdots,l\}$*) and* $\mathbf{y}_m^{(0,j)} \in \partial\boldsymbol{\Omega}_{(12,j)}$ *at time* t_m*,* $\mathbf{y}_{m\pm}^{(\alpha_j,j)} = \mathbf{y}_m^{(0,j)}$*. For any small* $\varepsilon > 0$*, there is a time interval* $[t_{m-\varepsilon}, t_m)$ *or* $(t_m, t_{m+\varepsilon}]$*. The constraint function* $\varphi_j(\mathbf{y}^{(\alpha_j)}, t, \boldsymbol{\lambda}_j)$ *is* $C^{r_{\alpha_j}}$ *continuous and* $|\varphi_j^{(r_{\alpha_j}+1)}(\mathbf{y}^{(\alpha_j,j)}, t, \boldsymbol{\lambda}_j)| < \infty$ *(*$r_{\alpha_j} \geq 2k_{\alpha_j}+1$*). For* $\mathbf{y}^{(\alpha_j,j)} \in \boldsymbol{\Omega}_{(\alpha_j,j)}$ *and* $\mathbf{y}^{(0,j)} \in \partial\boldsymbol{\Omega}_{(12,j)}$*, suppose* $D^{s_{\alpha_j}}\mathbb{F}^{(\alpha_j,j)}(\mathbf{y}^{(\alpha_j,j)}, t, \boldsymbol{\pi}^{(\alpha_j,j)}) = D^{s_{\alpha_j}}\mathbb{F}^{(0,j)}(\mathbf{y}^{(0,j)}, t, \boldsymbol{\lambda}_j)$ *(*$s_{\alpha_j} = 0,1,2,\cdots$*) for* $\mathbf{y}^{(\alpha_j,j)} = \mathbf{y}^{(0,j)}$*. The two dynamical systems in Eqs. (3.1) and (3.2) to l-constraints in Eq. (3.4) are desynchronized with l-dimensions for time* $t \in [t_{m_1}, t_{m_2}]$ *in the sense of Eq. (3.4) if and only if*

(i) *for all* $j \in \mathscr{L}$, $\mathbf{y}_m^{(\alpha_j,j)} \in \boldsymbol{\Omega}_{(\alpha_j,j)}$*, and* $\mathbf{y}_m^{(0,j)} \in \partial\boldsymbol{\Omega}_{(12,j)}$ *for any time* t_m

$$\mathbf{y}_m^{(\alpha_j,j)} = \mathbf{y}_m^{(0,j)}, \varphi_j^{(s_{\alpha_j})}(\mathbf{y}_m^{(\alpha_j,j)}, t_m, \boldsymbol{\lambda}_j) = 0$$
$$\text{for } \alpha_j = 1,2 \text{ and } s_{\alpha_j} = 0,1,2,\cdots \tag{4.58}$$

(ii) *for all* $j \in \mathscr{L}$, $\mathbf{y}_{\kappa}^{(\alpha_j,j)} \in \boldsymbol{\Omega}_{(\alpha_j,j)}^{+\varepsilon}$ *at time* $t_{\kappa}^{+} \in (t_m, t_{m+\varepsilon}]$ *and* $\mathbf{y}_m^{(0,j)} \in \partial\boldsymbol{\Omega}_{(12,j)}$ *with* $t_m \in (t_{m_1}, t_{m_2})$

$$\mathbf{y}_{\kappa}^{(\alpha_j,j)} \neq \mathbf{y}_m^{(0,j)}, \lim_{t_{\kappa}^{-}\to t_{m-}} \varphi_j^{(s_{\alpha_j})}(\mathbf{y}_{\kappa}^{(\alpha_j,j)}, t_{\kappa}^{+}, \boldsymbol{\lambda}_j) = 0 \ (s_{\alpha_j} = 1,2,\cdots,2k_{\alpha_j});$$
$$(-1)^{\alpha_j}\varphi_j^{(2k_{\alpha_j}+1)}(\mathbf{y}_{\kappa}^{(\alpha_j,j)}, t_{\kappa}^{+}, \boldsymbol{\lambda}_j) < 0,$$
$$\lim_{t_{\kappa}^{+}\to t_m} \varphi_j^{(2k_{\alpha_j}+1)}(\mathbf{y}_{\kappa}^{(\alpha_j,j)}, t_{\kappa}^{+}, \boldsymbol{\lambda}_j) = 0 \text{ for } \alpha_j = 1,2; \tag{4.59}$$

(iii) *for the* jth *constraint (*$j \in \mathscr{L}$*),* $\mathbf{y}_{\kappa}^{(\alpha_j,j)} \in \boldsymbol{\Omega}_{(\alpha_j,j)}^{-\varepsilon}$ *at time* $t_{\kappa}^{-} \in [t_{m-\varepsilon}, t_m)$ *and* $\mathbf{y}_m^{(0,j)} \in \partial\boldsymbol{\Omega}_{(12,j)}$ *with* $t_m \notin [t_{m_1}, t_{m_2}]$

$$\begin{aligned}&\mathbf{y}_{\kappa}^{(\alpha_j,j)} \neq \mathbf{y}_{m}^{(0,j)},\ \lim_{t_{\kappa}^{-}\to t_{m-}} \varphi_j^{(s_{\alpha_j})}(\mathbf{y}_{\kappa}^{(\alpha_j,j)}, t_{\kappa}^{-}, \boldsymbol{\lambda}_j) = 0\ (s_{\alpha_j} = 1, 2, \cdots, 2k_{\alpha_j});\\ &(-1)^{\alpha_j}\varphi_j^{(2k_{\alpha_j}+1)}(\mathbf{y}_{\kappa}^{(\alpha_j,j)}, t_{\kappa}^{-}, \boldsymbol{\lambda}_j) > 0 \text{ and}\\ &\lim_{t_{\kappa}^{-}\to t_m} \varphi_j^{(2k_{\alpha_j}+1)}(\mathbf{y}_{\kappa}^{(\alpha_j,j)}, t_{\kappa}^{-}, \boldsymbol{\lambda}_j) = 0 \text{ for } \alpha_j = 1, 2;\end{aligned} \tag{4.60}$$

(iv) *for the* jth *constraint* ($j \in \mathscr{L}$), $\mathbf{y}_{\kappa}^{(\alpha_j,j)} \in \boldsymbol{\Omega}_{(\alpha_j,j)}^{+\varepsilon}$ *at time* $t_{\kappa}^{-} \in [t_{m-\varepsilon}, t_{m-})$, $t_{\kappa}^{+} \in (t_{m+}, t_{m+\varepsilon}]$ *and* $\mathbf{y}_{m}^{(0,j)} \in \partial\boldsymbol{\Omega}_{(12,j)}$ *with* $t_m = t_{m_1}$ and t_{m_2}

$$\begin{aligned}&\mathbf{y}_{\kappa}^{(\alpha_j,j)} \neq \mathbf{y}_{m}^{(0,j)},\ \lim_{t_{\kappa}^{\pm}\to t_{m\pm}} \varphi_j^{(s_{\alpha_j})}(\mathbf{y}_{\kappa}^{(\alpha_j,j)}, t_{\kappa}^{\pm}, \boldsymbol{\lambda}_j) = 0\ (s_{\alpha_j} = 1, 2, \cdots, 2k_{\alpha_j}+1);\\ &\lim_{t\pm\to t_m} (-1)^{\alpha_j}\varphi_j^{(2k_{\alpha_j}+2)}(\mathbf{y}_{\kappa}^{(\alpha_j,j)}, t_{\kappa}^{\pm}, \boldsymbol{\lambda}_j) < 0 \text{ for } \alpha_j = 1, 2.\end{aligned} \tag{4.61}$$

Proof The proof is similar to the proof of Theorem 3.2 for each $j \in \mathscr{L}$. For all $j \in \mathscr{L}$, if the conditions in Eqs. (4.58) and (4.59) are satisfied, from Definition 4.14, the two dynamical systems in Eqs. (3.1) and (3.2) to constraints in Eq. (3.4) are synchronized for time $t \in (t_{m_1}, t_{m_2})$, vice versa. If the onset and vanishing conditions in Eqs. (4.60) and (4.61) hold, from Definition 4.18, the synchronization will start to form and to vanish, vice versa. The proof is completed. □

From the foregoing theorem, the desynchronization requires all the higher order derivatives of the constraint functions in Eq. (3.4) should be zero on the constraint surfaces and the highest order derivative of the constraint functions in domain should be greater than zero. In practical applications, such a condition is too strong for one to control the desynchronization of the two dynamical systems. Therefore, such a condition can be relaxed through a discontinuous vector field to the slave and master systems. Therefore, the following theorem for the desynchronization will be stated.

Theorem 4.7 *Consider two dynamical systems in Eqs. (3.1) and (3.2) with constraints in Eq. (3.4). For* $\mathbf{y}_{m\pm}^{(\alpha_j,j)} \in \boldsymbol{\Omega}_{(\alpha_j,j)}$ *(*$\alpha_j \in \mathscr{I}$ and $j \in \mathscr{L}$ *with* $\mathscr{I} = \{1, 2\}$ *and* $\mathscr{L} = \{1, 2, \cdots, l\}$*) and* $\mathbf{y}_{m}^{(0,j)} \in \partial\boldsymbol{\Omega}_{(12,j)}$ *at time* t_m, $\mathbf{y}_{m\pm}^{(\alpha_j,j)} = \mathbf{y}_{m}^{(0,j)}$. *For any small* $\varepsilon > 0$, *there is a time interval* $[t_{m-\varepsilon}, t_m)$ *or* $(t_m, t_{m+\varepsilon}]$. *The constraint function* $\varphi_j(\mathbf{y}^{(\alpha_j)}, t, \boldsymbol{\lambda}_j)$ *is* $C^{r_{\alpha_j}}$*-continuous and* $|\varphi_j^{(r_{\alpha_j}+1)}(\mathbf{y}^{(\alpha_j,j)}, t, \boldsymbol{\lambda}_j)| < \infty$ *(*$r_{\alpha_j} \geq 3$*). For* $\mathbf{y}^{(\alpha_j,j)} \in \boldsymbol{\Omega}_{(\alpha_j,j)}$ *and* $\mathbf{y}^{(0,j)} \in \partial\boldsymbol{\Omega}_{12(j)}$, $\mathbb{F}^{(\alpha_j,j)}(\mathbf{y}^{(\alpha_j,j)}, t, \boldsymbol{\pi}^{(\alpha_j,j)}) \neq \mathbb{F}^{(0,j)}(\mathbf{y}^{(0,j)}, t, \boldsymbol{\lambda}_j)$ *is assumed for* $\mathbf{y}^{(\alpha_j,j)} = \mathbf{y}^{(0,j)}$ *and* $\alpha_j \in \mathscr{I}$. *The two dynamical systems in Eqs. (3.1) and (3.2) to l-constraints in Eq. (3.4) are desynchronized with l-dimensions for time* $t \in [t_{m_1}, t_{m_2}]$ *if and only if*

(i) *for all* $j \in \mathscr{L}$, $\mathbf{y}_{m+}^{(\alpha_j,j)} = \mathbf{y}_{m}^{(0,j)}$, *and* $\mathbf{y}^{(\alpha_j,j)} \in \boldsymbol{\Omega}_{(\alpha_j,j)}$ *at time* $t_m \in [t_{m_1}, t_{m_2}]$ *(*$\alpha_j \in \mathscr{I}$*)*

$$\varphi_j(\mathbf{y}_{m+}^{(\alpha_j,j)}, t_{m+}, \boldsymbol{\lambda}_j) = \varphi_j(\mathbf{y}_{m}^{(0,j)}, t_m, \boldsymbol{\lambda}_j) = 0 \tag{4.62}$$

(ii) *for all* $j \in \mathscr{L}$, *time* $t_m \in (t_{m_1}, t_{m_2})$ *and* $\mathbf{y}_{m+}^{(\alpha_j,j)} = \mathbf{y}_m^{(0,j)}$ *with* $\alpha_j = 1, 2$

$$(-1)^{\alpha_j} \varphi_j^{(1)}(\mathbf{y}_{m+}^{(\alpha_j,j)}, t_{m+}, \boldsymbol{\lambda}_j) < 0 \tag{4.63}$$

(iii) *with the* jth *penetration for time* $t = t_{m_i}$, $\mathbf{y}_{m_i}^{(\alpha)} = \mathbf{y}_{m_i}^{(0)}$ *(*$i = 1, 2$*) for* $\alpha_j, \beta_j \in \mathscr{I}$ *and* $\beta_j \neq \alpha_j$

$$\left.\begin{array}{l} \varphi_j^{(1)}(\mathbf{y}_{m_i\pm}^{(\alpha_j,j)}, t_{m_i\pm}, \boldsymbol{\lambda}_j) = 0 \text{ and } (-1)^{\alpha_j} \varphi_j^{(2)}(\mathbf{y}_{m_i\pm}^{(\alpha_j,j)}, t_{m_i\pm}, \boldsymbol{\lambda}_j) < 0, \\ (-1)^{\beta_j} \varphi_j^{(1)}(\mathbf{y}_{m_i+}^{(\beta_j,j)}, t_{m_i+}, \boldsymbol{\lambda}_j) < 0; \end{array}\right\} \tag{4.64}$$

or with the jth *synchronization for time* $t = t_{m_i}$, $\mathbf{y}_{m_i}^{(\alpha)} = \mathbf{y}_{m_i}^{(0)}$ *(*$i = 1, 2$*) for* $\alpha_j, \beta_j \in \mathscr{I}$ *and* $\beta_j \neq \alpha_j$

$$\left.\begin{array}{l} \varphi_j^{(1)}(\mathbf{y}_{m_i\pm}^{(\alpha_j,j)}, t_{m_i\pm}, \boldsymbol{\lambda}_j) = 0 \text{ and } (-1)^{\alpha_j} \varphi_j^{(2)}(\mathbf{y}_{m_i\pm}^{(\alpha_j,j)}, t_{m_i\pm}, \boldsymbol{\lambda}_j) < 0, \\ \varphi_j^{(1)}(\mathbf{y}_{m_i\pm}^{(\beta_j,j)}, t_{m_i\pm}, \boldsymbol{\lambda}_j) = 0 \text{ and } (-1)^{\beta_j} \varphi_j^{(2)}(\mathbf{y}_{m_i\pm}^{(\beta_j,j)}, t_{m_i\pm}, \boldsymbol{\lambda}_j) < 0. \end{array}\right\} \tag{4.65}$$

Proof The proof is similar to the proof of Theorem 3.3 for each $j \in \mathscr{L}$. For all $j \in \mathscr{L}$, if the conditions in Eqs. (4.62) and (4.63) are satisfied, from Definition 4.5, the two dynamical systems in Eqs. (3.1) and (3.2) to constraints in Eq. (3.4) are desynchronized for time $t \in [t_{m_1}, t_{m_2}]$, vice versa. If the onset and vanishing conditions in Eqs. (4.64) and (4.65) hold, from Definition 4.9 or 4.10, the desynchronization will start to form and to vanish, vice versa. The proof is completed. □

In the foregoing theorem, the desynchronization of two dynamical systems to multiple constraints is without any singularity except for the onset and vanishing condition. If the desynchronization of two dynamical systems to multiple constraints possesses higher-order singularity, the following theorem is presented.

Theorem 4.8 *Consider two dynamical systems in Eqs. (3.1) and (3.2) with constraints in Eq. (3.4). For* $\mathbf{y}_{m\pm}^{(\alpha_j,j)} \in \boldsymbol{\Omega}_{(\alpha_j,j)}$ ($\alpha_j \in \mathscr{I}$ and $j \in \mathscr{L}$ *with* $\mathscr{I} = \{1, 2\}$ *and* $\mathscr{L} = \{1, 2, \cdots, l\}$*) and* $\mathbf{y}_m^{(0,j)} \in \partial\boldsymbol{\Omega}_{(12,j)}$ *at time* t_m, $\mathbf{y}_{m\pm}^{(\alpha_j,j)} = \mathbf{y}_m^{(0,j)}$. *For any small* $\varepsilon > 0$, *there is a time interval* $[t_{m-\varepsilon}, t_m)$ *or* $(t_m, t_{m+\varepsilon}]$. *The constraint function* $\varphi_j(\mathbf{y}^{(\alpha_j)}, t, \boldsymbol{\lambda}_j)$ *is* $C^{r_{\alpha_j}}$*-continuous and* $|\varphi_j^{(r_{\alpha_j}+1)}(\mathbf{y}^{(\alpha_j,j)}, t, \boldsymbol{\lambda}_j)| < \infty$ *(*$r_{\alpha_j} \geq 2k_{\alpha_j} + 1$*). For* $\mathbf{y}^{(\alpha_j,j)} \in \boldsymbol{\Omega}_{(\alpha_j,j)}$ *and* $\mathbf{y}^{(0,j)} \in \partial\boldsymbol{\Omega}_{12(j)}$, $\mathbb{F}^{(\alpha_j,j)}(\mathbf{y}^{(\alpha_j,j)}, t, \boldsymbol{\pi}^{(\alpha_j,j)}) \neq \mathbb{F}^{(0,j)}(\mathbf{y}^{(0,j)}, t, \boldsymbol{\lambda}_j)$ *is assumed for* $\mathbf{y}^{(\alpha_j,j)} = \mathbf{y}^{(0,j)}$ *and* $\alpha_j \in \mathscr{I}$. *The two dynamical systems in Eqs. (3.1) and (3.2) to l-constraints in Eq. (3.4) are desynchronized of the* $(2\mathbf{k}_\alpha : 2\mathbf{k}_\beta)$*-type with l-dimensions for time* $t \in [t_{m_1}, t_{m_2}]$ *if and only if*

(i) *for all* $j \in \mathscr{L}$, $\mathbf{y}_{m+}^{(\alpha_j,j)} = \mathbf{y}_m^{(0,j)}$, *and* $\mathbf{y}^{(\alpha_j,j)} \in \boldsymbol{\Omega}_{(\alpha_j,j)}$ *at time* $t_m \in [t_{m_1}, t_{m_2}]$ ($\alpha_j \in \mathscr{I}$)

$$\varphi_j(\mathbf{y}_{m+}^{(\alpha_j,j)}, t_{m+}, \boldsymbol{\lambda}_j) = \varphi_j(\mathbf{y}_m^{(0,j)}, t_m, \boldsymbol{\lambda}_j) = 0 \tag{4.66}$$

(ii) *for all* $j \in \mathscr{L}$, *time* $t_m \in (t_{m_1}, t_{m_2})$ *and* $\mathbf{y}_{m+}^{(\alpha_j,j)} = \mathbf{y}_m^{(0,j)}$ *with* $\alpha_j = 1, 2$

$$\begin{aligned} &\varphi_j^{(s_{\alpha_j})}(\mathbf{y}_{m+}^{(\alpha_j,j)}, t_{m+}, \boldsymbol{\lambda}_j) = 0 \ (s_{\alpha_j} = 1, 2, \cdots, 2k_{\alpha_j}), \\ &(-1)^{\alpha_j}\varphi_j^{(2k_{\alpha_j}+1)}(\mathbf{y}_{m+}^{(\alpha_j,j)}, t_{m+}, \boldsymbol{\lambda}_j) < 0 \end{aligned} \tag{4.67}$$

(iii) *with the* jth *penetration of the* $(2k_{\alpha_j} : 2k_{\beta_j})$*-type for time* $t = t_{m_i}$, $\mathbf{y}_{m_i}^{(\alpha)} = \mathbf{y}_{m_i}^{(0)}$ $(i = 1, 2)$ *for* $\alpha_j, \beta_j \in \mathscr{I}$ and $\beta_j \neq \alpha_j$

$$\left.\begin{aligned} &\varphi_j^{(s_{\alpha_j})}(\mathbf{y}_{m_i\pm}^{(\alpha_j,j)}, t_{m_i\pm}, \boldsymbol{\lambda}_j) = 0 \ (s_{\alpha_j} = 1, 2, \cdots 2k_{\alpha_j} + 1) \\ &(-1)^{\alpha_j}\varphi^{(2k_{\alpha_j}+2)}(\mathbf{y}_{m_i\pm}^{(\alpha_j,j)}, t_{m_i\pm}, \boldsymbol{\lambda}_j) < 0; \\ &\varphi_j^{(s_{\beta_j})}(\mathbf{y}_{m_i+}^{(\beta_j,j)}, t_{m_i+}, \boldsymbol{\lambda}_j) = 0 \ (s_{\beta_j} = 1, 2, \cdots 2k_{\beta_j}) \\ &(-1)^{\beta_j}\varphi_j^{(2k_{\beta_j}+1)}(\mathbf{y}_{m_i+}^{(\beta_j,j)}, t_{m_i+}, \boldsymbol{\lambda}_j) < 0; \end{aligned}\right\} \tag{4.68}$$

or with the jth *synchronization of the* $(2k_{\alpha_j} : 2k_{\beta_j})$*-type for time* $t = t_{m_i}$, $\mathbf{y}_{m_i}^{(\alpha)} = \mathbf{y}_{m_i}^{(0)}$ $(i = 1, 2)$ *for* $\alpha_j, \beta_j \in \mathscr{I}$ *and* $\beta_j \neq \alpha_j$

$$\begin{aligned} &\varphi_j^{(s_{\alpha_j})}(\mathbf{y}_{m_i\pm}^{(\alpha_j,j)}, t_{m_i\pm}, \boldsymbol{\lambda}_j) = 0 \ (s_{\alpha_j} = 1, 2, \cdots 2k_{\alpha_j} + 1), \\ &(-1)^{\alpha_j}\varphi^{(2k_{\alpha_j}+2)}(\mathbf{y}_{m_i\pm}^{(\alpha_j,j)}, t_{m_i\pm}, \boldsymbol{\lambda}_j) < 0; \\ &\varphi_j^{(s_{\beta_j})}(\mathbf{y}_{m_i\pm}^{(\beta_j,j)}, t_{m_i\pm}, \boldsymbol{\lambda}_j) = 0 \ (s_{\beta_j} = 1, 2, \cdots 2k_{\beta_j} + 1), \\ &(-1)^{\beta_j}\varphi_j^{(2k_{\beta_j}+2)}(\mathbf{y}_{m_i\pm}^{(\beta_j,j)}, t_{m_i\pm}, \boldsymbol{\lambda}_j) < 0. \end{aligned} \tag{4.69}$$

Proof The proof is similar to the proof of Theorem 3.4 for each $j \in \mathscr{L}$. For all $j \in \mathscr{L}$, if the conditions in Eqs. (4.66) and (4.67) are satisfied, from Definition 4.14, the two dynamical systems in Eqs. (3.1) and (3.2) to constraints in Eq. (3.4) are synchronized with the $(2k_{\alpha_j} : 2k_{\beta_j})$-type for time $t \in [t_{m_1}, t_{m_2}]$, vice versa. If the onset and vanishing conditions in Eqs. (4.68) and (4.69) hold, from Definition 4.17 or 4.18, the desynchronization will start to form and to vanish, vice versa. The proof is completed. □

4.7 Penetration to All Constraints

If $\mathbb{F}^{(\alpha_j,j)}(\mathbf{y}^{(\alpha_j,j)}, t, \boldsymbol{\pi}^{(\alpha_j,j)}) = \mathbb{F}^{(0,j)}(\mathbf{y}^{(0,j)}, t, \boldsymbol{\lambda}_j)$ for $\mathbf{y}^{(\alpha_j,j)} = \mathbf{y}^{(0,j)}$ and $\alpha_j \in \{1, 2\}$ with all $j \in \mathscr{L}$, the two dynamical systems to multiple constraints do not have any penetration. For such a penetration, $\mathbb{F}^{(\alpha_j,j)}(\mathbf{y}^{(\alpha_j,j)}, t, \boldsymbol{\pi}^{(\alpha_j,j)}) \neq \mathbb{F}^{(0,j)}(\mathbf{y}^{(0,j)}, t, \boldsymbol{\lambda}_j)$ should exist. Thus, the corresponding conditions for the l-dimensional penetration of two dynamical systems with l-constraints are presented through the following theorems.

Theorem 4.9 *Consider two dynamical systems in Eqs. (3.1) and (3.2) with constraints in Eq. (3.4). For* $\mathbf{y}_{m\pm}^{(\alpha_j,j)} \in \boldsymbol{\Omega}_{(\alpha_j,j)}$ *(*$\alpha_j \in \mathscr{I}$ and $j \in \mathscr{L}$ *with* $\mathscr{I} = \{1,2\}$ *and* $\mathscr{L} = \{1,2,\cdots,l\}$*) and* $\mathbf{y}_m^{(0,j)} \in \partial\boldsymbol{\Omega}_{(12,j)}$ *at time* t_m*,* $\mathbf{y}_{m\pm}^{(\alpha_j,j)} = \mathbf{y}_m^{(0,j)}$*. For any small* $\varepsilon > 0$*, there is a time interval* $[t_{m-\varepsilon}, t_m)$ *or* $(t_m, t_{m+\varepsilon}]$*. The constraint function* $\varphi_j(\mathbf{y}^{(\alpha_j)}, t, \boldsymbol{\lambda}_j)$ *is* $C^{r_{\alpha_j}}$ *continuous and* $|\varphi_j^{(r_{\alpha_j}+1)}(\mathbf{y}^{(\alpha_j,j)}, t, \boldsymbol{\lambda}_j)| < \infty$ *(*$r_{\alpha_j} \geq 3$*). For* $\mathbf{y}^{(\alpha_j,j)} \in \boldsymbol{\Omega}_{(\alpha_j,j)}$ *and* $\mathbf{y}^{(0,j)} \in \partial\boldsymbol{\Omega}_{12(j)}$*,* $\mathbb{F}^{(\alpha_j,j)}(\mathbf{y}^{(\alpha_j,j)}, t, \boldsymbol{\pi}^{(\alpha_j,j)}) \neq \mathbb{F}^{(0,j)}(\mathbf{y}^{(0,j)}, t, \boldsymbol{\lambda}_j)$ *is assumed for* $\mathbf{y}^{(\alpha_j,j)} = \mathbf{y}^{(0,j)}$ *and* $\alpha_j \in \mathscr{I}$*. The two dynamical systems in Eqs. (3.1) and (3.2) to l-constraints in Eq. (3.4) are penetrated with l-dimensions for time* $t \in [t_{m_1}, t_{m_2}]$ *if and only if*

(i) *for all* $j \in \mathscr{L}$*, time* $t = t_m \in (t_{m_1}, t_{m_2})$*,* $\mathbf{y}_{m-}^{(\alpha_j,j)} = \mathbf{y}_m^{(0)} = \mathbf{y}_{m+}^{(\beta_j,j)}$

$$(-1)^{\alpha_j}\varphi_j^{(1)}(\mathbf{y}_{m-}^{(\alpha_j,j)}, t_{m-}, \boldsymbol{\lambda}_j) > 0 \text{ and } (-1)^{\beta_j}\varphi_j^{(1)}(\mathbf{y}_{m+}^{(\beta_j,j)}, t_{m+}, \boldsymbol{\lambda}_j) < 0 \tag{4.70}$$

(ii) *with the synchronization to the j*th *constraint for time* $t = t_{m_i}$*,* $\mathbf{y}_{m-}^{(\alpha_j,j)} = \mathbf{y}_m^{(0)} = \mathbf{y}_{m\pm}^{(\beta_j,j)}$ *(*$i = 1,2$*),*

$$\begin{aligned} &(-1)^{\alpha_j}\varphi_j^{(1)}(\mathbf{y}_{m-}^{(\alpha_j,j)}, t_{m-}, \boldsymbol{\lambda}_j) > 0; \\ &(-1)^{\beta_j}\varphi_j^{(1)}(\mathbf{y}_{m\pm}^{(\beta_j,j)}, t_{m\pm}, \boldsymbol{\lambda}_j) = 0 \text{ and } (-1)^{\beta_j}\varphi^{(2)}(\mathbf{y}_{m\pm}^{(\beta_j,j)}, t_{m\pm}, \boldsymbol{\lambda}_j) < 0. \end{aligned} \tag{4.71}$$

or with the desynchronization to the jth constraint only for time $t = t_{m_i}$*,* $\mathbf{y}_{m\mp}^{(\alpha_j,j)} = \mathbf{y}_m^{(0)} = \mathbf{y}_{m+}^{(\beta_j,j)}$ *(*$i = 1,2$*),*

$$\begin{aligned} &\varphi_j^{(1)}(\mathbf{y}_{m_i\mp}^{(\alpha_j,j)}, t_{m_i\mp}, \boldsymbol{\lambda}_j) = 0 \text{ and } (-1)^{\alpha_j}\varphi_j^{(2)}(\mathbf{y}_{m_i\mp}^{(\alpha_j,j)}, t_{m_i\mp}, \boldsymbol{\lambda}_j) < 0, \\ &(-1)^{\beta_j}\varphi_j^{(1)}(\mathbf{y}_{m+}^{(\beta_j,j)}, t_{m+}, \boldsymbol{\lambda}_j) < 0; \end{aligned} \tag{4.72}$$

or with the switching penetration to the jth constraint only for time $t = t_{m_i}$*,* $\mathbf{y}_{m\mp}^{(\alpha_j,j)} = \mathbf{y}_m^{(0)} = \mathbf{y}_{m\pm}^{(\beta_j,j)}$ *(*$i = 1,2$*),*

$$\begin{aligned} &\varphi_j^{(1)}(\mathbf{y}_{m_i\mp}^{(\alpha_j,j)}, t_{m_i\mp}, \boldsymbol{\lambda}_j) = 0 \text{ and } (-1)^{\alpha_j}\varphi_j^{(2)}(\mathbf{y}_{m_i\mp}^{(\alpha_j,j)}, t_{m_i\mp}, \boldsymbol{\lambda}_j) < 0, \\ &(-1)^{\beta_j}\varphi_j^{(1)}(\mathbf{y}_{m\pm}^{(\beta_j,j)}, t_{m\pm}, \boldsymbol{\lambda}_j) = 0 \text{ and } (-1)^{\beta_j}\varphi_j^{(2)}(\mathbf{y}_{m\pm}^{(\beta_j,j)}, t_{m\pm}, \boldsymbol{\lambda}_j) < 0. \end{aligned} \tag{4.73}$$

Proof The proof is similar to the proof of Theorem 3.3 for each $j \in \mathscr{L}$. For all $j \in \mathscr{L}$, if the conditions in Eq. (4.70) are satisfied, the two dynamical systems in Eqs. (3.1) and (3.2) to constraints in Eq. (3.4) are penetrated with l-dimensions for time $t \in [t_{m_1}, t_{m_2}]$, vice versa. If the onset and vanishing conditions in Eqs. (4.71)–(4.73) are satisfied, the penetration of the two dynamical systems with l- constraints will start to be formed or to vanish, vice versa. This theorem is proved. □

Theorem 4.10 *Consider two dynamical systems in Eqs. (3.1) and (3.2) with constraints in Eq. (3.4). For* $\mathbf{y}_{m\pm}^{(\alpha_j,j)} \in \boldsymbol{\Omega}_{(\alpha_j,j)}$ *(*$\alpha_j \in \mathscr{I}$ and $j \in \mathscr{L}$ *with* $\mathscr{I} = \{1,2\}$ *and* $\mathscr{L} = \{1,2,\cdots,l\}$*) and* $\mathbf{y}_m^{(0,j)} \in \partial\boldsymbol{\Omega}_{(12,j)}$ *at time* t_m, $\mathbf{y}_{m\pm}^{(\alpha_j,j)} = \mathbf{y}_m^{(0,j)}$*. For any small* $\varepsilon > 0$, *there is a time interval* $[t_{m-\varepsilon}, t_m)$ *or* $(t_m, t_{m+\varepsilon}]$*. The constraint function* $\varphi_j(\mathbf{y}^{(\alpha_j)}, t, \boldsymbol{\lambda}_j)$ *is* $C^{r_{\alpha_j}}$*-continuous and* $|\varphi_j^{(r_{\alpha_j}+1)}(\mathbf{y}^{(\alpha_j,j)}, t, \boldsymbol{\lambda}_j)| < \infty$ *(*$r_{\alpha_j} \geq 2k_{\alpha_j} + 1$*). For* $\mathbf{y}^{(\alpha_j,j)} \in \boldsymbol{\Omega}_{(\alpha_j,j)}$ *and* $\mathbf{y}^{(0,j)} \in \partial\boldsymbol{\Omega}_{12(j)}$, $\mathbb{F}^{(\alpha_j,j)}(\mathbf{y}^{(\alpha_j,j)}, t, \boldsymbol{\pi}^{(\alpha_j,j)}) \neq \mathbb{F}^{(0,j)}(\mathbf{y}^{(0,j)}, t, \boldsymbol{\lambda}_j)$ *is assumed for* $\mathbf{y}^{(\alpha_j,j)} = \mathbf{y}^{(0,j)}$ *and* $\alpha_j \in \mathscr{I}$*. The two dynamical systems in Eqs. (3.1) and (3.2) to l-constraints in Eq. (3.4) are penetrated of the* $(2\mathbf{k}_\alpha : 2\mathbf{k}_\beta)$*-type with l-dimensions for time* $t \in [t_{m_1}, t_{m_2}]$ *if and only if*

(i) *for all* $j \in \mathscr{L}$, *time* $t = t_m \in (t_{m_1}, t_{m_2})$, $\mathbf{y}_{m-}^{(\alpha_j,j)} = \mathbf{y}_m^{(0)} = \mathbf{y}_{m+}^{(\beta_j,j)}$

$$\left.\begin{aligned}
&\varphi_j^{(s_{\alpha_j})}(\mathbf{y}_{m-}^{(\alpha_j,j)}, t_{m-}, \boldsymbol{\lambda}_j) > 0 \ (s_{\alpha_j} = 1,2,\cdots,2k_{\alpha_j}),\\
&(-1)^{\alpha_j}\varphi_j^{(2k_{\alpha_j}+1)}(\mathbf{y}_{m-}^{(\alpha_j,j)}, t_{m-}, \boldsymbol{\lambda}_j) > 0;\\
&\varphi_j^{(s_{\beta_j})}(\mathbf{y}_{m+}^{(\beta_j,j)}, t_{m+}, \boldsymbol{\lambda}_j) = 0 \ (s_{\beta_j} = 1,2,\cdots,2k_{\beta_j}),\\
&(-1)^{\beta_j}\varphi_j^{(2k_{\beta_j}+1)}(\mathbf{y}_{m+}^{(\beta_j,j)}, t_{m+}, \boldsymbol{\lambda}_j) < 0;
\end{aligned}\right\} \tag{4.74}$$

(ii) *with the synchronization of the* $(2k_{\alpha_j} : 2k_{\beta_j})$*-type to the* jth *constraint for time* $t = t_{m_i}$, $\mathbf{y}_{m-}^{(\alpha_j,j)} = \mathbf{y}_m^{(0)} = \mathbf{y}_{m\pm}^{(\beta_j,j)}$ *(*$i = 1,2$*),*

$$\left.\begin{aligned}
&\varphi_j^{(s_{\alpha_j})}(\mathbf{y}_{m-}^{(\alpha_j,j)}, t_{m-}, \boldsymbol{\lambda}_j) = 0 \ (s_{\alpha_j} = 1,2,\cdots,2k_{\alpha_j}),\\
&(-1)^{\alpha_j}\varphi_j^{(2k_{\alpha_j}+1)}(\mathbf{y}_{m-}^{(\alpha_j,j)}, t_{m-}, \boldsymbol{\lambda}_j) > 0;
\end{aligned}\right\} \tag{4.75a}$$

$$\left.\begin{aligned}
&\varphi_j^{(s_{\beta_j})}(\mathbf{y}_{m\pm}^{(\beta_j,j)}, t_{m\pm}, \boldsymbol{\lambda}_j) = 0 \ (s_{\beta_j} = 1,2,\cdots,2k_{\beta_j}+1),\\
&(-1)^{\beta_j}\varphi_j^{(2k_{\beta_j}+2)}(\mathbf{y}_{m\pm}^{(\beta_j,j)}, t_{m\pm}, \boldsymbol{\lambda}_j) < 0;
\end{aligned}\right\} \tag{4.75b}$$

or with the desynchronization of the $(2k_{\alpha_j} : 2k_{\beta_j})$*-type to the* jth *constraint only for time* $t = t_{m_i}$, $\mathbf{y}_{m\mp}^{(\alpha_j,j)} = \mathbf{y}_m^{(0)} = \mathbf{y}_{m+}^{(\beta_j,j)}$ *(*$i = 1,2$*),*

$$\left.\begin{aligned}
&\varphi_j^{(s_{\alpha_j})}(\mathbf{y}_{m\mp}^{(\alpha_j,j)}, t_{m\mp}, \boldsymbol{\lambda}_j) = 0 \ (s_{\alpha_j} = 1,2,\cdots,2k_{\alpha_j}+1),\\
&(-1)^{\alpha_j}\varphi_j^{(2k_{\alpha_j}+2)}(\mathbf{y}_{m\mp}^{(\alpha_j,j)}, t_{m\mp}, \boldsymbol{\lambda}_j) < 0;\\
&\varphi_j^{(s_{\beta_j})}(\mathbf{y}_{m+}^{(\beta_j,j)}, t_{m+}, \boldsymbol{\lambda}_j) = 0 \ (s_{\beta_j} = 1,2,\cdots,2k_{\beta_j}),\\
&(-1)^{\beta_j}\varphi_j^{(2k_{\beta_j}+1)}(\mathbf{y}_{m+}^{(\beta_j,j)}, t_{m+}, \boldsymbol{\lambda}_j) < 0;
\end{aligned}\right\} \tag{4.76}$$

or with the switching penetration of the $(2k_{\beta_j} : 2k_{\alpha_j})$*-type to the* jth *constraint only for time* $t = t_{m_i}$, $\mathbf{y}_{m\mp}^{(\alpha_j,j)} = \mathbf{y}_m^{(0)} = \mathbf{y}_{m\pm}^{(\beta_j,j)}$ $(i = 1, 2)$,

$$\begin{aligned}
&\varphi_j^{(s_{\alpha_j})}(\mathbf{y}_{m\mp}^{(\alpha_j,j)}, t_{m\mp}, \boldsymbol{\lambda}_j) = 0 \ (s_{\alpha_j} = 1, 2, \cdots, 2k_{\alpha_j} + 1),\\
&(-1)^{\alpha_j}\varphi_j^{(2k_{\alpha_j}+2)}(\mathbf{y}_{m\mp}^{(\alpha_j,j)}, t_{m\mp}, \boldsymbol{\lambda}_j) < 0;\\
&\varphi_j^{(s_{\beta_j})}(\mathbf{y}_{m\pm}^{(\beta_j,j)}, t_{m\pm}, \boldsymbol{\lambda}_j) = 0 \ (s_{\beta_j} = 1, 2, \cdots, 2k_{\beta_j} + 1),\\
&(-1)^{\beta_j}\varphi_j^{(2k_{\beta_j}+2)}(\mathbf{y}_{m\pm}^{(\beta_j,j)}, t_{m\pm}, \boldsymbol{\lambda}_j) < 0.
\end{aligned} \tag{4.77}$$

Proof The proof is similar to the proof of Theorem 3.4 for each $j \in \mathscr{L}$. For all $j \in \mathscr{L}$, if the conditions are satisfied in Eq. (4.74), the two dynamical systems in Eqs. (3.1) and (3.2) to constraints in Eq. (3.4) are penetrated of the $(2\mathbf{k}_\alpha : 2\mathbf{k}_\beta)$-type with l-dimensions for $t \in (t_{m_1}, t_{m_2})$, vice versa. If the switching conditions for the synchronization– penetration, desynchronization–penetration, and penetration–penetration in Eqs. (4.75)–(4.77) are satisfied, the onset and vanishing of the $(2\mathbf{k}_\alpha : 2\mathbf{k}_\beta)$-penetration with l-dimensions occur, vice versa. This theorem is proved. □

4.8 Synchronization–Desynchronization–Penetration

As in Luo [2], from the theory of discontinuous dynamical systems, in this section, the mixture of the synchronization, desynchronization, and penetration to multiple constraints is discussed.

Theorem 4.11 *Consider two dynamical systems in Eqs. (3.1) and (3.2) with constraints in Eq. (3.4). For* $\mathbf{y}_{m\pm}^{(\alpha_j,j)} \in \boldsymbol{\Omega}_{(\alpha_j,j)}$ ($\alpha_j \in \mathscr{I}$ and $j \in \mathscr{L}$ *with* $\mathscr{I} = \{1, 2\}$ *and* $\mathscr{L} = \{1, 2, \cdots, l\}$) *and* $\mathbf{y}_m^{(0,j)} \in \partial\boldsymbol{\Omega}_{(12,j)}$ *at time* t_m, $\mathbf{y}_{m\pm}^{(\alpha_j,j)} = \mathbf{y}_m^{(0,j)}$. *For any small* $\varepsilon > 0$, *there is a time interval* $[t_{m-\varepsilon}, t_m)$ *or* $(t_m, t_{m+\varepsilon}]$. *The constraint function* $\varphi_j(\mathbf{y}^{(\alpha_j)}, t, \boldsymbol{\lambda}_j)$ *is* $C^{r_{\alpha_j}}$ *continuous and* $|\varphi_j^{(r_{\alpha_j}+1)}(\mathbf{y}^{(\alpha_j,j)}, t, \boldsymbol{\lambda}_j)| < \infty$ ($r_{\alpha_j} \geq 3$). *For* $\mathbf{y}^{(\alpha_j,j)} \in \boldsymbol{\Omega}_{(\alpha_j,j)}$ *and* $\mathbf{y}^{(0,j)} \in \partial\boldsymbol{\Omega}_{(12,j)}$, *suppose* $D^{s_{\alpha_j}}\mathbb{F}^{(\alpha_j,j)}(\mathbf{y}^{(\alpha_j,j)}, t, \boldsymbol{\pi}^{(\alpha_j,j)}) = D^{s_{\alpha_j}}\mathbb{F}^{(0,j)}(\mathbf{y}^{(0,j)}, t, \boldsymbol{\lambda}_j)$ ($s_{\alpha_j} = 0,\ 1, 2, \cdots$) *for* $\mathbf{y}^{(\alpha_j,j)} = \mathbf{y}^{(0,j)}$. *The two dynamical systems in Eqs. (3.1) and (3.2) to l-constraints in Eq. (3.4) are of the* (l_1, l_2)*-synchronization and desynchronization for time* $[t_{m_1}, t_{m_2}]$ *if and only if*

(i) *for all* $j \in \mathscr{L}$, $\mathbf{y}_m^{(\alpha_j,j)} \in \boldsymbol{\Omega}_{(\alpha_j,j)}$ *and* $\mathbf{y}_m^{(0,j)} \in \partial\boldsymbol{\Omega}_{(12,j)}$ *for any time* t_m

$$\begin{aligned}
&\mathbf{y}_m^{(\alpha_j,j)} = \mathbf{y}_m^{(0,j)}, \varphi_j^{(s_{\alpha_j})}(\mathbf{y}_m^{(\alpha_j,j)}, t_m, \boldsymbol{\lambda}_j) = 0\\
&\text{for } \alpha_j = 1, 2 \text{ and } s_{\alpha_j} = 0, 1, 2, \cdots
\end{aligned} \tag{4.78}$$

(ii) *for all* $j_1 \in \mathscr{L}_1$ *and* $\alpha_{j_1} = 1, 2$

$$\begin{aligned}&\mathbf{y}_{\kappa}^{(\alpha_{j_1},j_1)} \neq \mathbf{y}_m^{(0,j_1)}, (-1)^{\alpha_{j_1}}\varphi_{j_1}^{(1)}(\mathbf{y}_{\kappa}^{(\alpha_{j_1},j_1)}, t_{\kappa}^{-}, \boldsymbol{\lambda}_{j_1}) > 0 \text{ and}\\&\lim_{t_{\kappa}^{-}\to t_m} \varphi_{j_1}^{(1)}(\mathbf{y}_{\kappa}^{(\alpha_{j_1},j_1)}, t_{\kappa}^{-}, \boldsymbol{\lambda}_{j_1}) = 0 \text{ for } \alpha_j = 1,2\end{aligned} \tag{4.79}$$

with $\mathbf{y}_{\kappa_1}^{(\alpha_{j_1},j_1)} \in \boldsymbol{\Omega}_{(\alpha_{j_1},j_1)}^{-\varepsilon}$ *at time* $t_{\kappa}^{-} \in [t_{m-\varepsilon}, t_m)$ *and* $\mathbf{y}_m^{(0,j_1)} \in \partial\boldsymbol{\Omega}_{(12,j_1)}$ *for* $t_m \in (t_{m_1}, t_{m_2})$;

(iii) *for all* $j_2 \in \mathscr{L}_2$ *and* $\alpha_{j_2} = 1,2$

$$\begin{aligned}&\mathbf{y}_{\kappa}^{(\alpha_{j_2},j_2)} \neq \mathbf{y}_m^{(0,j_2)}, (-1)^{\alpha_{j_2}}\varphi_{j_2}^{(1)}(\mathbf{y}_{\kappa}^{(\alpha_{j_2},j_2)}, t_{\kappa}^{+}, \boldsymbol{\lambda}_{j_2}) < 0 \text{ and}\\&\lim_{t_{\kappa}^{+}\to t_m} \varphi_{j_2}^{(1)}(\mathbf{y}_{\kappa}^{(\alpha_{j_2},j_2)}, t_{\kappa}^{+}, \boldsymbol{\lambda}_{j_2}) = 0 \text{ for } \alpha_{j_2} = 1,2\end{aligned} \tag{4.80}$$

with $\mathbf{y}_{\kappa}^{(\alpha_{j_2},j_2)} \in \boldsymbol{\Omega}_{(\alpha_{j_2},j_2)}^{+\varepsilon}$ *at time* $t_{\kappa}^{+} \in (t_m, t_{m+\varepsilon}]$ *and* $\mathbf{y}_m^{(0,j_2)} \in \partial\boldsymbol{\Omega}_{(12,j_2)}$ *for* $t_m \in (t_{m_1}, t_{m_2})$;

(iv) *for some* $j_1 \in \mathscr{L}_1$

$$\begin{aligned}&\mathbf{y}_{\kappa}^{(\alpha_{j_1},j_1)} \neq \mathbf{y}_m^{(0,j_1)}, (-1)^{\alpha_{j_1}}\varphi_{j_1}^{(1)}(\mathbf{y}_{\kappa}^{(\alpha_{j_1},j_1)}, t_{\kappa}^{+}, \boldsymbol{\lambda}_{j_1}) < 0 \text{ and}\\&\lim_{t_{\kappa}^{+}\to t_m} \varphi_{j_1}^{(1)}(\mathbf{y}_{\kappa}^{(\alpha_{j_1},j_1)}, t_{\kappa}^{+}, \boldsymbol{\lambda}_{j_1}) = 0 \text{ for } \alpha_j = 1,2\end{aligned} \tag{4.81}$$

with $\mathbf{y}_{\kappa}^{(\alpha_{j_1},j_1)} \in \boldsymbol{\Omega}_{(\alpha_{j_1},j_1)}^{+\varepsilon}$ *at time* $t_{\kappa} \in (t_m, t_{m+\varepsilon}]$ *and* $\mathbf{y}_m^{(0,j_1)} \in \partial\boldsymbol{\Omega}_{(12,j_1)}$ *for* $t_m \notin [t_{m_1}, t_{m_2}]$; *or for some* $j_2 \in \mathscr{L}_2$

$$\begin{aligned}&\mathbf{y}_{\kappa}^{(\alpha_{j_2},j_2)} \neq \mathbf{y}_m^{(0,j_2)}, (-1)^{\alpha_{j_2}}\varphi_{j_2}^{(1)}(\mathbf{y}_{\kappa}^{(\alpha_{j_2},j_2)}, t_{\kappa}^{-}, \boldsymbol{\lambda}_{j_2}) > 0 \text{ and}\\&\lim_{t_{\kappa}^{-}\to t_m} \varphi_{j_2}^{(1)}(\mathbf{y}_{\kappa}^{(\alpha_{j_2},j_2)}, t_{\kappa}^{-}, \boldsymbol{\lambda}_{j_2}) = 0 \text{ for } \alpha_{j_2} = 1,2\end{aligned} \tag{4.82}$$

with $\mathbf{y}_{\kappa}^{(\alpha_{j_2},j_2)} \in \boldsymbol{\Omega}_{(\alpha_{j_2},j_2)}^{-\varepsilon}$ *at time* $t_{\kappa} \in [t_{m-\varepsilon}, t_m)$ *and* $\mathbf{y}_m^{(0,j_2)} \in \partial\boldsymbol{\Omega}_{(12,j_2)}$ *for* $t_m \notin [t_{m_1}, t_{m_2}]$;

(v) *for* $j \in \{j_1, j_2\}$ *in (iv) and* $\alpha_j = 1,2$

$$\begin{aligned}&\mathbf{y}_{\kappa}^{(\alpha)} \neq \mathbf{y}_m^{(0)}, \lim_{t_{\kappa}^{\pm}\to t_{m\pm}} \varphi_j^{(1)}(\mathbf{y}_{\kappa}^{(\alpha)}, t_{\kappa}^{\pm}, \boldsymbol{\lambda}_j) = 0,\\&\lim_{t_{\kappa}^{\pm}\to t_{m\pm}} (-1)^{\alpha_j}\varphi_j^{(2)}(\mathbf{y}_{\kappa}^{(\alpha)}, t_{\kappa}^{\pm}, \boldsymbol{\lambda}_j) < 0 \text{ for } \alpha_j = 1,2\end{aligned} \tag{4.83}$$

with $\mathbf{y}_{\kappa}^{(\alpha_j,j)} \in \boldsymbol{\Omega}_{(\alpha_j,j)}^{\pm\varepsilon}$ *at time* $t_{\kappa}^{-} \in [t_{m-\varepsilon}, t_{m-}), t_{\kappa}^{+} \in (t_{m+}, t_{m+\varepsilon}]$ *and* $\mathbf{y}_m^{(0,j)} \in \partial\boldsymbol{\Omega}_{(12,j)}$ *for* $t_m = t_{m_1}$ and t_{m_2}.

Proof The proof is similar to the proof of Theorem 3.1 for each $j \in \mathscr{L}$. For all $j \in \mathscr{L}$, if the conditions are satisfied, the two dynamical systems in Eqs. (3.1) and (3.2) to l-constraints in Eq. (3.4) are of the (l_1, l_2)-synchronization and desynchronization for time $[t_{m_1}, t_{m_2}]$, vice versa. This theorem is proved. □

Theorem 4.12 *Consider two dynamical systems in Eqs. (3.1) and (3.2) with constraints in Eq. (3.4). For* $\mathbf{y}_m^{(\alpha_j,j)} \in \boldsymbol{\Omega}_{(\alpha_j,j)}$ *(*$\alpha_j \in \mathscr{I}$ and $j \in \mathscr{L}$ *with* $\mathscr{I} = \{1,2\}$ *and* $\mathscr{L} = \{1,2,\cdots,l\}$*) and* $\mathbf{y}_m^{(0,j)} \in \partial\boldsymbol{\Omega}_{(12,j)}$ *at time* t_m, $\mathbf{y}_m^{(\alpha_j,j)} = \mathbf{y}_m^{(0,j)}$. *For any small* $\varepsilon > 0$, *there is a time interval* $[t_{m-\varepsilon}, t_m)$ *or* $(t_m, t_{m+\varepsilon}]$. *The constraint function* $\varphi(\mathbf{y}^{(\alpha_j)}, t, \boldsymbol{\lambda})$ *is* $C^{r_{\alpha_j}}$*-continuous and* $|\varphi_j^{(r_{\alpha_j}+1)}(\mathbf{y}^{(\alpha_j,j)}, t, \boldsymbol{\lambda}_j)| < \infty$ *(*$r_{\alpha_j} \geq 2k_{\alpha_j}+1$*). For* $\mathbf{y}^{(\alpha_j,j)} \in \boldsymbol{\Omega}_{(\alpha_j,j)}$ *and* $\mathbf{y}^{(0,j)} \in \partial\boldsymbol{\Omega}_{(12,j)}$, *suppose* $D^{s_{\alpha_j}}\mathbb{F}^{(\alpha_j,j)}(\mathbf{y}^{(\alpha_j,j)}, t, \boldsymbol{\pi}^{(\alpha_j,j)}) = D^{s_{\alpha_j}}\mathbb{F}^{(0,j)}(\mathbf{y}^{(0,j)}, t, \boldsymbol{\lambda}_j)$ *(*$s_{\alpha_j} = 0, 1, 2, \cdots$*) for the two dynamical systems in Eqs. (3.1) and (3.2) to l-constraints in Eq. (3.4) are of an* (l_1, l_2)-$(2\mathbf{k}_{\alpha 1}+1)$th *synchronization and* $(2\mathbf{k}_{\alpha 2}+1)$th-*desynchronization for time* $[t_{m_1}, t_{m_2}]$ *if and only if*

(i) *for all* $j \in \mathscr{L}$, $\mathbf{y}_m^{(\alpha_j,j)} \in \boldsymbol{\Omega}_{(\alpha_j,j)}$ *and* $\mathbf{y}_m^{(0,j)} \in \partial\boldsymbol{\Omega}_{(12,j)}$ *for any time* t_m

$$\begin{aligned}&\mathbf{y}_m^{(\alpha_j,j)} = \mathbf{y}_m^{(0,j)}, \varphi_j^{(s_{\alpha_j})}(\mathbf{y}_m^{(\alpha_j,j)}, t_m, \boldsymbol{\lambda}_j) = D^{s_{\alpha_j}}\varphi_j(\mathbf{y}_m^{(\alpha_j,j)}, t_m, \boldsymbol{\lambda}_j) = 0\\&\text{for } \alpha_j = 1,2 \text{ and } s_{\alpha_j} = 0,1,2,\cdots\end{aligned} \tag{4.84}$$

(ii) *for all* $j_1 \in \mathscr{L}_1$ *and* $\alpha_{j_1} = 1,2$

$$\begin{aligned}&\mathbf{y}_\kappa^{(\alpha_{j_1},j_1)} \neq \mathbf{y}_m^{(0,j_1)},\\&\lim_{t_\kappa^- \to t_{m-}} \varphi_{j_1}^{(s_{\alpha_{j_1}})}(\mathbf{y}_\kappa^{(\alpha_{j_1},j_1)}, t_\kappa^-, \boldsymbol{\lambda}_{j_1}) = 0 \ (s_{\alpha_{j_1}} = 1,2,\cdots,2k_{\alpha_{j_1}});\\&(-1)^{\alpha_{j_1}}\varphi_{j_1}^{(2k_{\alpha_{j_1}}+1)}(\mathbf{y}_\kappa^{(\alpha_{j_1},j_1)}, t_\kappa^-, \boldsymbol{\lambda}_{j_1}) > 0 \text{ and}\\&\lim_{t_\kappa^- \to t_m} \varphi_{j_1}^{(2k_{\alpha_{j_1}}+1)}(\mathbf{y}_\kappa^{(\alpha_{j_1},j_1)}, t_\kappa^-, \boldsymbol{\lambda}_{j_1}) = 0 \text{ for } \alpha_j = 1,2;\end{aligned} \tag{4.85}$$

with $\mathbf{y}_{\kappa 1}^{(\alpha_{j_1},j_1)} \in \boldsymbol{\Omega}_{(\alpha_{j_1},j_1)}^{-\varepsilon}$ *at time* $t_\kappa^- \in [t_{m-\varepsilon}, t_m)$ *and* $\mathbf{y}_m^{(0,j_1)} \in \partial\boldsymbol{\Omega}_{(12,j_1)}$ *for* $t_m \in (t_{m_1}, t_{m_2})$;

(iii) *for all* $j_2 \in \mathscr{L}_2$ *and* $\alpha_{j_2} = 1,2$

$$\begin{aligned}&\mathbf{y}_\kappa^{(\alpha_{j_2},j_2)} \neq \mathbf{y}_m^{(0,j_2)},\\&\lim_{t_\kappa^+ \to t_{m+}} \varphi_{j_2}^{(s_{\alpha_{j_2}})}(\mathbf{y}_\kappa^{(\alpha_{j_2},j_2)}, t_\kappa^+, \boldsymbol{\lambda}_{j_2}) = 0 \ (s_{\alpha_{j_2}} = 1,2,\cdots,2k_{\alpha_{j_2}});\\&(-1)^{\alpha_{j_2}}\varphi_{j_2}^{(2k_{\alpha_{j_2}}+1)}(\mathbf{y}_\kappa^{(\alpha_{j_2},j_2)}, t_\kappa^+, \boldsymbol{\lambda}_{j_2}) < 0 \text{ and}\\&\lim_{t_\kappa^+ \to t_m} \varphi_{j_2}^{(2k_{\alpha_{j_2}}+1)}(\mathbf{y}_\kappa^{(\alpha_{j_2},j_2)}, t_\kappa^+, \boldsymbol{\lambda}_{j_2}) = 0;\end{aligned} \tag{4.86}$$

with $\mathbf{y}_\kappa^{(\alpha_{j_2},j_2)} \in \boldsymbol{\Omega}_{(\alpha_{j_2},j_2)}^{+\varepsilon}$ *at time* $t_\kappa^+ \in (t_m, t_{m+\varepsilon}]$ *and* $\mathbf{y}_m^{(0,j_2)} \in \partial\boldsymbol{\Omega}_{(12,j_2)}$ *for* $t_m \in (t_{m_1}, t_{m_2})$;

(iv) *for some* $j_1 \in \mathscr{L}_1$ *and* $\alpha_{j_1} = 1,2$

$$\begin{aligned}
&\mathbf{y}_{\kappa}^{(\alpha_{j_1},j_1)} \neq \mathbf{y}_{m}^{(0,j_1)},\\
&\lim_{t_\kappa^+\to t_{m+}} \varphi_{j_1}^{(s_{\alpha_{j_1}})}(\mathbf{y}_{\kappa}^{(\alpha_{j_1},j_1)},t_\kappa^+,\boldsymbol{\lambda}_{j_1}) = 0 \ (s_{\alpha_{j_1}}=1,2,\cdots,2k_{\alpha_{j_1}});\\
&(-1)^{\alpha_{j_1}}\varphi_{j}^{(2k_{\alpha_{j_1}}+1)}(\mathbf{y}_{\kappa}^{(\alpha_{j_1},j_1)},t_\kappa^+,\boldsymbol{\lambda}_{j_1})<0 \text{ and}\\
&\lim_{t_\kappa^+\to t_{m}} \varphi_{j_1}^{(2k_{\alpha_{j_1}}+1)}(\mathbf{y}_{\kappa}^{(\alpha_{j_1},j_1)},t_\kappa^+,\boldsymbol{\lambda}_{j_1}) = 0;
\end{aligned} \tag{4.87}$$

with $\mathbf{y}_{\kappa}^{(\alpha_{j_1},j_1)} \in \boldsymbol{\Omega}_{(\alpha_{j_1},j_1)}^{+\varepsilon}$ *at time* $t_\kappa \in (t_m, t_{m+\varepsilon}]$ *and* $\mathbf{y}_m^{(0,j)} \in \partial\boldsymbol{\Omega}_{(12,j)}$ *for* $t_m \notin [t_{m_1}, t_{m_2}]$; *or for some* $j_2 \in \mathscr{L}_2$ *and* $\alpha_{j_2} = 1,2$

$$\begin{aligned}
&\mathbf{y}_{\kappa}^{(\alpha_{j_2},j_2)} \neq \mathbf{y}_{m}^{(0,j_2)},\\
&\lim_{t_\kappa^-\to t_{m+}} \varphi_{j_2}^{(s_{\alpha_{j_2}})}(\mathbf{y}_{\kappa}^{(\alpha_{j_2},j_2)},t_\kappa^-,\boldsymbol{\lambda}_{j_2}) = 0 \ (s_{\alpha_{j_2}}=1,2,\cdots,2k_{\alpha_{j_2}});\\
&(-1)^{\alpha_{j_2}}\varphi_{j_2}^{(2k_{\alpha_{j_2}}+1)}(\mathbf{y}_{\kappa}^{(\alpha_{j_2},j_2)},t_\kappa^-,\boldsymbol{\lambda}_{j_2})>0 \text{ and}\\
&\lim_{t_\kappa^-\to t_{m}} \varphi_{j_2}^{(2k_{\alpha_{j_2}}+1)}(\mathbf{y}_{\kappa}^{(\alpha_{j_2},j_2)},t_\kappa^-,\boldsymbol{\lambda}_{j_2}) = 0;
\end{aligned} \tag{4.88}$$

with $\mathbf{y}_{\kappa}^{(\alpha_{j_2},j_2)} \in \boldsymbol{\Omega}_{(\alpha_{j_2},j_2)}^{-\varepsilon}$ *at time* $t_\kappa \in (t_m, t_{m+\varepsilon}]$ *and* $\mathbf{y}_m^{(0,j)} \in \partial\boldsymbol{\Omega}_{(12,j)}$ *for* $t_m \notin [t_{m_1}, t_{m_2}]$;

(v) *for* $j \in \{j_1, j_2\}$ *in (iv) and* $\alpha_j = 1,2$

$$\begin{aligned}
&\mathbf{y}_{\kappa}^{(\alpha)} \neq \mathbf{y}_{m}^{(0)},\\
&\lim_{t_\kappa^\pm\to t_{m\pm}} \varphi_{j}^{(s_{\alpha_j})}(\mathbf{y}_{\kappa}^{(\alpha_j,j)},t_\kappa^\pm,\boldsymbol{\lambda}_{j}) = 0 \ (s_{\alpha_j}=1,2,\cdots,2k_{\alpha_j}+1);\\
&\lim_{t_\kappa^\pm\to t_{m\pm}} (-1)^{\alpha_j}\varphi_{j}^{(2k_{\alpha_j}+2)}(\mathbf{y}_{\kappa}^{(\alpha_j,j)},t_\kappa^\pm,\boldsymbol{\lambda}_{j})<0.
\end{aligned} \tag{4.89}$$

with $\mathbf{y}_{\kappa}^{(\alpha_j,j)} \in \boldsymbol{\Omega}_{(\alpha_j,j)}^{+\varepsilon}$ *at time* $t_\kappa^- \in [t_{m-\varepsilon}, t_{m-})$, $t_\kappa^+ \in (t_{m+}, t_{m+\varepsilon}]$ *and* $\mathbf{y}_m^{(0,j)} \in \partial\boldsymbol{\Omega}_{(12,j)}$ *for* $t_m = t_{m_1}$ and t_{m_2}.

Proof The proof is similar to the proof of Theorem 3.2 for each $j \in \mathscr{L}$. For all $j \in \mathscr{L}$, if the conditions are satisfied, the two dynamical systems in Eqs. (3.1) and (3.2) to constraints in Eq. (3.4) are of an (l_1, l_2)-$(2\mathbf{k}_{\alpha 1}+1)$th -synchronization and $(2\mathbf{k}_{\alpha 2}+1)$th-desynchronization, vice versa. This theorem is proved. □

Theorem 4.13 *Consider two dynamical systems in Eqs. (3.1) and (3.2) with constraints in Eq. (3.4). For* $\mathbf{y}_{m\pm}^{(\alpha_j,j)} \in \boldsymbol{\Omega}_{(\alpha_j,j)}$ *(*$\alpha_j \in \mathscr{I}$ *and* $j \in \mathscr{L}$ *with* $\mathscr{I} = \{1,2\}$ *and* $\mathscr{L} = \{1,2,\cdots,l\}$*) and* $\mathbf{y}_m^{(0,j)} \in \partial\boldsymbol{\Omega}_{(12,j)}$ *at time* t_m, $\mathbf{y}_{m\pm}^{(\alpha_j,j)} = \mathbf{y}_m^{(0,j)}$. *For any small* $\varepsilon > 0$, *there is a time interval* $[t_{m-\varepsilon}, t_m)$ *or* $(t_m, t_{m+\varepsilon}]$. *The constraint function* $\varphi_j(\mathbf{y}^{(\alpha_j)}, t, \boldsymbol{\lambda}_j)$ *is* $C^{r_{\alpha_j}}$*-continuous and* $|\varphi_j^{(r_{\alpha_j}+1)}(\mathbf{y}^{(\alpha_j,j)}, t, \boldsymbol{\lambda}_j)| < \infty$ *(*$r_{\alpha_j} \geq 3$*). For* $\mathbf{y}^{(\alpha_j,j)} \in \boldsymbol{\Omega}_{(\alpha_j,j)}$ *and* $\mathbf{y}^{(0,j)} \in \partial\boldsymbol{\Omega}_{12(j)}$, $\mathbb{F}^{(\alpha_j,j)}(\mathbf{y}^{(\alpha_j,j)}, t, \boldsymbol{\pi}^{(\alpha_j,j)}) \neq \mathbb{F}^{(0,j)}(\mathbf{y}^{(0,j)}, t, \boldsymbol{\lambda}_j)$ *is assumed for* $\mathbf{y}^{(\alpha_j,j)} = \mathbf{y}^{(0,j)}$

and $\alpha_j \in \mathscr{I}$. *The two dynamical systems in Eqs. (3.1) and (3.2) to l-constraints in Eq. (3.4) are of the* (l_1, l_2, l_3)*-synchronization, desynchronization, and penetration for time* $t \in [t_{m_1}, t_{m_2}]$ *if and only if*

(i) *for* $\mathbf{y}_{m\pm}^{(\alpha_j, j)} = \mathbf{y}_m^{(0,j)}$ *and* $\mathbf{y}^{(\alpha_j, j)} \in \mathbf{\Omega}_{(\alpha_j, j)}$ ($\alpha_j \in \mathscr{I}$) *at time* $t = t_m \in [t_{m_1}, t_{m_2}]$

$$\begin{gathered}\varphi_j(\mathbf{y}_{m\pm}^{(\alpha_j,j)}, t_{m\pm}, \boldsymbol{\lambda}_j) = \varphi_j(\mathbf{y}_m^{(0,j)}, t_m, \boldsymbol{\lambda}_j) = 0 \\ \text{for all } j \in \mathscr{L} = \mathscr{L}_1 \cup \mathscr{L}_2 \cup \mathscr{L}_3 \end{gathered} \tag{4.90}$$

(ii) *for all* $j_1 \in \mathscr{L}_1$, $\mathbf{y}_{m-}^{(\alpha_{j_1}, j_1)} = \mathbf{y}_m^{(0, j_1)} = \mathbf{y}_{m-}^{(\beta_{j_1}, j_1)}$ *with* $\alpha_{j_1}, \beta_{j_1} \in \mathscr{I}$ *at time* $t_m \in (t_{m_1}, t_{m_2})$

$$\begin{aligned}(-1)^{\alpha_{j_1}} \varphi_{j_1}^{(1)}(\mathbf{y}_{m-}^{(\alpha_{j_1}, j_1)}, t_{m-}, \boldsymbol{\lambda}_{j_1}) > 0, \\ (-1)^{\beta_{j_1}} \varphi_{j_1}^{(1)}(\mathbf{y}_{m-}^{(\beta_{j_1}, j_1)}, t_{m-}, \boldsymbol{\lambda}_{j_1}) > 0;\end{aligned} \tag{4.91}$$

(iii) *for all* $j_2 \in \mathscr{L}_2$, $\mathbf{y}_{m+}^{(\alpha_{j_2}, j_2)} = \mathbf{y}_m^{(0, j_2)} = \mathbf{y}_{m+}^{(\beta_{j_2}, j_2)}$ *at time* $t_m \in (t_{m_1}, t_{m_2})$

$$\begin{gathered}(-1)^{\alpha_{j_2}} \varphi_{j_2}^{(1)}(\mathbf{y}_{m+}^{(\alpha_{j_2}, j_2)}, t_{m+}, \boldsymbol{\lambda}_{j_1}) < 0, \\ (-1)^{\beta_{j_2}} \varphi_{j_2}^{(1)}(\mathbf{y}_{m+}^{(\beta_{j_2}, j_2)}, t_{m+}, \boldsymbol{\lambda}_{j_2}) < 0 \\ \text{for } \alpha_{j_2}, \beta_{j_2} \in \mathscr{I} \text{ and } \alpha_{j_2} \neq \beta_{j_2};\end{gathered} \tag{4.92}$$

(iv) *for all* $j_3 \in \mathscr{L}_3$, $\mathbf{y}_{m-}^{(\alpha_{j_3}, j_3)} = \mathbf{y}_m^{(0, j_3)} = \mathbf{y}_{m+}^{(\beta_{j_3}, j_3)}$ *at time* $t_m \in (t_{m_1}, t_{m_2})$

$$\begin{gathered}(-1)^{\alpha_{j_3}} \varphi_{j_3}^{(1)}(\mathbf{y}_{m-}^{(\alpha_{j_3}, j_3)}, t_{m-}, \boldsymbol{\lambda}_{j_3}) < 0, \\ (-1)^{\beta_{j_3}} \varphi_{j_3}^{(1)}(\mathbf{y}_{m+}^{(\beta_{j_3}, j_3)}, t_{m+}, \boldsymbol{\lambda}_{j_3}) < 0 \\ \text{for } \alpha_{j_3}, \beta_{j_3} \in \mathscr{I} \text{ and } \alpha_{j_3} \neq \beta_{j_3};\end{gathered} \tag{4.93}$$

(v) *for one of* $j \in \{j_1, j_2, j_3\}$ *with time* $t = t_{m_i}$, $\mathbf{y}_{m_i}^{(\alpha_j, j)} = \mathbf{y}_{m_i}^{(0,j)}$ $(i = 1, 2)$, $\alpha_j \in \{1,2\}$

$$\begin{aligned}\varphi_j^{(1)}(\mathbf{y}_{m_i\pm}^{(\alpha_j,j)}, t_{m_i\pm}, \boldsymbol{\lambda}_j) = 0, \\ (-1)^{\alpha_j} \varphi_j^{(2)}(\mathbf{y}_{m_i\pm}^{(\alpha_j,j)}, t_{m_i\pm}, \boldsymbol{\lambda}_j) < 0,\end{aligned} \tag{4.94}$$

and/or for $\beta_j \in \{1,2\}$

$$\begin{aligned}\varphi_j^{(1)}(\mathbf{y}_{m_i\pm}^{(\beta_j,j)}, t_{m_i\pm}, \boldsymbol{\lambda}_j) = 0, \\ (-1)^{\beta_j} \varphi_j^{(2)}(\mathbf{y}_{m_i\pm}^{(\beta_j,j)}, t_{m_i\pm}, \boldsymbol{\lambda}_j) < 0.\end{aligned} \tag{4.95}$$

Proof The proof is similar to the proof of Theorem 3.3 for each $j \in \mathscr{L}$. For all $j \in \mathscr{L}$, if the conditions are satisfied, of the (l_1, l_2, l_3) -synchronization, desynchronization, and penetration for time $t \in [t_{m_1}, t_{m_2}]$, vice versa. This theorem is proved. □

Theorem 4.14 *Consider two dynamical systems in Eqs. (3.1) and (3.2) with constraints in Eq. (3.4). For* $\mathbf{y}_{m\pm}^{(\alpha_j,j)} \in \mathbf{\Omega}_{(\alpha_j,j)}$ ($\alpha_j \in \mathscr{I}$ *and* $j \in \mathscr{L}$ *with* $\mathscr{I} = \{1,2\}$ *and* $\mathscr{L} = \{1,2,\cdots,l\}$) *and* $\mathbf{y}_m^{(0,j)} \in \partial\mathbf{\Omega}_{(12,j)}$ *at time* t_m, $\mathbf{y}_{m\pm}^{(\alpha_j,j)} = \mathbf{y}_m^{(0,j)}$. *For any small* $\varepsilon > 0$, *there is a time interval* $[t_{m-\varepsilon}, t_m)$ *or* $(t_m, t_{m+\varepsilon}]$. *The constraint function* $\varphi_j(\mathbf{y}^{(\alpha_j)}, t, \boldsymbol{\lambda}_j)$ *is* $C^{r_{\alpha_j}}$ *continuous and* $|\varphi_j^{(r_{\alpha_j}+1)}(\mathbf{y}^{(\alpha_j,j)}, t, \boldsymbol{\lambda}_j)| < \infty$ ($r_{\alpha_j} \geq 2k_{\alpha_j} + 1$). *For* $\mathbf{y}^{(\alpha_j,j)} \in \mathbf{\Omega}_{(\alpha_j,j)}$ *and* $\mathbf{y}^{(0,j)} \in \partial\mathbf{\Omega}_{12(j)}$, $\mathbb{F}^{(\alpha_j,j)}(\mathbf{y}^{(\alpha_j,j)}, t, \boldsymbol{\pi}^{(\alpha_j,j)}) \neq \mathbb{F}^{(0,j)}(\mathbf{y}^{(0,j)}, t, \boldsymbol{\lambda}_j)$ *is assumed for* $\mathbf{y}^{(\alpha_j,j)} = \mathbf{y}^{(0,j)}$ *and* $\alpha_j \in \mathscr{I}$. *The two dynamical systems in* Eqs. (3.1) and (3.2) to l-constraints in Eq. (3.4) are of the (l_1, l_2, l_3)-synchronization, desynchronization, and penetration of the $\cup_{i=1}^{3}(2\mathbf{k}_{\alpha i} : 2\mathbf{k}_{\beta i})$ -type for time $t \in [t_{m_1}, t_{m_2}]$ if and only if

(i) for $\mathbf{y}_{m\pm}^{(\alpha_j,j)} = \mathbf{y}_m^{(0,j)}$ and $\mathbf{y}^{(\alpha_j,j)} \in \mathbf{\Omega}_{(\alpha_j,j)}$ ($\alpha_j \in \mathscr{I}$) at time $t = t_m \in [t_{m_1}, t_{m_2}]$

$$\left.\begin{array}{l}\varphi_j(\mathbf{y}_{m\pm}^{(\alpha_j,j)}, t_{m\pm}, \boldsymbol{\lambda}_j) = \varphi_j(\mathbf{y}_m^{(0,j)}, t_m, \boldsymbol{\lambda}_j) = 0\\ \text{for all } j \in \mathscr{L} = \mathscr{L}_1 \cup \mathscr{L}_2 \cup \mathscr{L}_3\end{array}\right\} \tag{4.96}$$

(ii) *for all* $j_1 \in \mathscr{L}_1$, $\mathbf{y}_{m-}^{(\alpha_{j_1},j_1)} = \mathbf{y}_m^{(0,j_1)}$ *and* $\alpha_{j_1} = 1,2$ *at time* $t_m \in (t_{m_1}, t_{m_2})$

$$\left.\begin{array}{l}\varphi_{j_1}^{(s_{\alpha_{j_1}})}(\mathbf{y}_{m-}^{(\alpha_{j_1},j_1)}, t_{m-}, \boldsymbol{\lambda}_{j_1}) = 0 \text{ for } s_{\alpha_{j_1}} = 1,2,\cdots,2k_{\alpha_{j_1}}\\ (-1)^{\alpha_{j_1}}\varphi_{j_1}^{(2k_{\alpha_{j_1}}+1)}(\mathbf{y}_{m-}^{(\alpha_{j_1},j_1)}, t_{m-}, \boldsymbol{\lambda}_{j_1}) > 0,\end{array}\right\} \tag{4.97}$$

(iii) *for all* $j_2 \in \mathscr{L}_2$, $\mathbf{y}_{m+}^{(\alpha_{j_2},j_2)} = \mathbf{y}_m^{(0,j_2)} = \mathbf{y}_{m+}^{(\beta_{j_2},j_2)}$ *at time* $t_m \in (t_{m_1}, t_{m_2})$

$$\left.\begin{array}{l}\varphi_{j_2}^{(s_{\alpha_{j_2}})}(\mathbf{y}_{m+}^{(\alpha_{j_2},j_2)}, t_{m+}, \boldsymbol{\lambda}_{j_1}) = 0 \text{ for } s_{\alpha_{j_2}} = 1,2,\cdots,2k_{\alpha_{j_2}}\\ (-1)^{\alpha_{j_2}}\varphi_{j_2}^{(2k_{\alpha_{j_2}}+1)}(\mathbf{y}_{m+}^{(\alpha_{j_2},j_2)}, t_{m+}, \boldsymbol{\lambda}_{j_1}) < 0 \text{ for } \alpha_{j_2} = 1,2;\end{array}\right\} \tag{4.98}$$

(iv) *for all* $j_3 \in \mathscr{L}_3$, $\mathbf{y}_{m-}^{(\alpha_{j_3},j_3)} = \mathbf{y}_m^{(0,j_3)} = \mathbf{y}_{m+}^{(\beta_{j_3},j_3)}$ *at time* $t_m \in (t_{m_1}, t_{m_2})$

$$\left.\begin{array}{l}\varphi_{j_3}^{(s_{\alpha_{j_3}})}(\mathbf{y}_{m-}^{(\alpha_{j_3},j_3)}, t_{m-}, \boldsymbol{\lambda}_{j_3}) = 0 \text{ for } s_{\alpha_{j_3}} = 1,2,\cdots,2k_{\alpha_{j_3}},\\ (-1)^{\alpha_{j_3}}\varphi_{j_3}^{(2k_{\alpha_{j_3}}+1)}(\mathbf{y}_{m-}^{(\alpha_{j_3},j_3)}, t_{m-}, \boldsymbol{\lambda}_{j_3}) > 0;\\ \varphi_{j_3}^{(s_{\beta_{j_3}})}(\mathbf{y}_{m+}^{(\beta_{j_3},j_3)}, t_{m+}, \boldsymbol{\lambda}_{j_3}) = 0 \text{ for } s_{\beta_{j_3}} = 1,2,\cdots,2k_{\beta_{j_3}},\\ (-1)^{\beta_{j_1}}\varphi_{j_3}^{(2k_{\beta_{j_3}}+1)}(\mathbf{y}_{m+}^{(\beta_{j_3},j_3)}, t_{m+}, \boldsymbol{\lambda}_{j_3}) < 0\\ \text{for } \alpha_{j_3}, \beta_{j_3} \in \mathscr{I} \text{ and } \alpha_{j_3} \neq \beta_{j_3};\end{array}\right\} \tag{4.99}$$

(v) *for one of* $j \in \{j_1, j_2, j_3\}$ *with the* $(2k_{\alpha_j} : 2k_{\beta_j})$*-singularity with time* $t = t_{m_i}$, $\mathbf{y}_{m_i}^{(\alpha_j, j)} = \mathbf{y}_{m_i}^{(0,j)}$ *(*$i = 1, 2$*) and* $\alpha_j \in \{1, 2\}$

$$\begin{aligned} &\varphi_j^{(s_{\alpha_j})}(\mathbf{y}_{m_i\pm}^{(\alpha_j,j)}, t_{m_i\pm}, \boldsymbol{\lambda}_j) = 0 \ (s_{\alpha_j} = 1, 2, \cdots 2k_{\alpha_j} + 1), \\ &(-1)^{\alpha_j} \varphi_j^{(2k_{\alpha_j}+2)}(\mathbf{y}_{m_i\pm}^{(\alpha_j,j)}, t_{m_i\pm}, \boldsymbol{\lambda}_j) < 0; \end{aligned} \tag{4.100}$$

and/or for $\beta_j \in \{1, 2\}$

$$\begin{aligned} &\varphi_j^{(s_{\beta_j})}(\mathbf{y}_{m_i\pm}^{(\beta_j,j)}, t_{m_i\pm}, \boldsymbol{\lambda}_j) = 0 \ (s_{\beta_j} = 1, 2, \cdots 2k_{\beta_j} + 1), \\ &(-1)^{\beta_j} \varphi_j^{(2k_\beta+2)}(\mathbf{y}_{m_i\pm}^{(\beta_j,j)}, t_{m_i\pm}, \boldsymbol{\lambda}_j) < 0. \end{aligned} \tag{4.101}$$

Proof The proof is similar to the proof of Theorem 3.4 for each $j \in \mathscr{L}$. For all $j \in \mathscr{L}$, if the conditions are satisfied, two dynamical systems in Eqs. (3.1) and (3.2) to l-constraints in Eq. (3.4) are of the (l_1, l_2, l_3)-synchronization, desynchronization, and penetration of the $\cup_{i=1}^{3} (2\mathbf{k}_{\alpha i} : 2\mathbf{k}_{\beta i})$-type for time $t \in [t_{m_1}, t_{m_2}]$, vice versa. This theorem is proved. □

4.9 Complexity by System Synchronization

To discuss the synchronization complexity, consider many master systems and many slave systems. A few master and slave systems with constraints can be synchronized.

Definition 4.21 A $\mathcal{S}$-set of slave systems is defined as

$$\mathcal{S} \equiv \left\{ {}^{(I_s)}\mathbf{x} = {}^{(I_s)}\mathbf{F}({}^{(I_s)}\mathbf{x}, t, {}^{(I_s)}\mathbf{p}) \middle| I_s = 1, 2, \cdots; {}^{(I_s)}\mathbf{x} \in \mathscr{R}^{n(I_s)}; {}^{(I_s)}\mathbf{p} \in \mathscr{R}^{k(I_s)} \right\} \tag{4.102}$$

and an $\mathcal{M}$-set of master systems is defined as

$$\mathcal{M} \equiv \left\{ {}^{(I_r)}\dot{\mathbf{x}} = {}^{(I_r)}\mathbf{F}({}^{(I_r)}\mathbf{x}, t, {}^{(I_r)}\mathbf{p}) \middle| I_r = 1, 2, \cdots; {}^{(I_r)}\mathbf{x} \in \mathscr{R}^{n(I_r)}; {}^{(I_r)}\mathbf{p} \in \mathscr{R}^{k(I_r)} \right\} \tag{4.103}$$

This definition gives a cluster of slave systems and a cluster of master systems. To investigate the synchronization of the slave and master systems, the slave and master systems can be selected from such $\mathcal{S}$-set of slave systems and $\mathcal{M}$-set of master systems. For a slave system in the $\mathcal{S}$-set of slave systems, it can be synchronized with many master systems in the $\mathcal{M}$-set of master systems with the corresponding constraints. The constraints for such synchronization can be either single or multiple constraints, and the synchronized components for such constraints can be either full or partial components from those slave and master systems. Based on this reason, the subspace set in state space should be defined.

Definition 4.22 A subspace set of the I_sth-slave system is defined as

$$\Omega_{\mathcal{S}} \equiv \left\{ {}^{(I_s,\mu_s)}\Omega_{\mathcal{S}} \middle| \mu_s = 1, 2, \cdots; I_s = 1, 2, \cdots \right\}, \tag{4.104}$$

where

$$\begin{aligned} {}^{(I_s,\mu)}\Omega_{\mathcal{S}} \equiv \Big\{ & {}^{(I_s,\mu_s)}\mathbf{x} \Big| {}^{(I_s,\mu_s)}\mathbf{x} = \left({}^{(I_s,\mu_s)}x_1, {}^{(I_s,\mu_s)}x_2, \cdots, {}^{(I_s,\mu_s)}x_{\Gamma_\mu} \right)^{\mathrm{T}}, \\ & \mu_s = 1, 2, \cdots, \text{ and } \Gamma_{\mu_s} \le n_{(I_s)}, \\ & {}^{(I_s,\mu_s)}x_i \in \left\{ {}^{(I_s)}x_1, {}^{(I_s)}x_2, \cdots, {}^{(I_s)}x_{n_{(I_s)}} \right\}, i = 1, 2, \cdots, \Gamma_{\mu_s} \Big\} \end{aligned} \tag{4.105}$$

and a subspace set of the I_rth-master system is defined as

$$\Omega_{\mathcal{M}} \equiv \left\{ {}^{(I_r,\mu_r)}\Omega_{\mathcal{M}} \middle| \mu_r = 1, 2, \cdots; I_r = 1, 2, \cdots \right\} \tag{4.106}$$

where

$$\begin{aligned} {}^{(I_r,\mu_r)}\Omega_{\mathcal{M}} \equiv \Big\{ & {}^{(I_r,\mu_r)}\mathbf{x} \Big| {}^{(I_r,\mu_r)}\mathbf{x} = \left({}^{(I_r,\mu_r)}x_1, {}^{(I_r,\mu_r)}x_2, \cdots, {}^{(I_r,\mu_r)}x_{\Gamma_{\mu_r}} \right)^{T}, \\ & \mu_r = 1, 2, \cdots, \text{ and } \Gamma_{\mu_r} \le n_{(I_r)}, \\ & {}^{(I_r,\mu_r)}x_i \in \left\{ {}^{(I_r)}x_1, {}^{(I_r)}X_2, \cdots, {}^{(I_r)}X_{n_{(Ir)}} \right\}, i = 1, 2, \cdots, \Gamma_{\mu_r} \Big\} \end{aligned} \tag{4.107}$$

From the foregoing definitions of the two subspace sets for slave and master systems, each subspace for the I_sth-slave system (or the I_rth-master system) is arbitrarily selected from $n_{(I_s)}$-components (or $n_{(I_r)}$-components). Based on such phase subspaces for the I_sth-slave system and the I_rth-master system, the corresponding constraint can be defined for the synchronization of such slave and master systems on the two subspaces. Thus, the corresponding $\mathcal{C}$-set of the constraints for the slave and master systems is defined as follows.

Definition 4.23 For two subspaces ${}^{(I_s,\mu_s)}\Omega_{\mathcal{S}}$ and ${}^{(I_r,\mu_r)}\Omega_{\mathcal{M}}$, a $\mathcal{C}$-set of constraints is defined as

$$\mathcal{C} \equiv \left\{ {}^{(I_r,I_s)}\mathcal{C} \middle| I_r = 1, 2, \cdots; I_s = 1, 2, \cdots \right\} \tag{4.108}$$

where

$$
\begin{aligned}
{}^{(I_r,I_s)}\mathscr{C} \equiv \Big\{ & {}^{(I_r,I_s)}\varphi_j({}^{(I_r,\mu_s)}\mathbf{x},\, {}^{(I_s,\mu_s)}\mathbf{x}, t, \boldsymbol{\lambda}_j) = 0 \big| j = 1, 2, \cdots; \\
& \mu_r, \mu_s = 1, 2, \cdots; \boldsymbol{\lambda}_j \in \mathscr{R}^{n_j} \Big\}.
\end{aligned}
\tag{4.109}
$$

Definition 4.24 Consider M_s-slave systems from the $\mathscr{S}$-set of slave systems and M_r-master systems from the $\mathscr{M}$-set of master systems

$$
{}^{(I_s)}\dot{\mathbf{x}} = {}^{(I_s)}\mathbf{F}({}^{(I_s)}\mathbf{x}, t, {}^{(I_s)}\mathbf{p}) \text{ for all } I_s \in \{1, 2, \cdots, M_s\} \tag{4.110}
$$

$$
{}^{(I_r)}\dot{\mathbf{x}} = {}^{(I_r)}\mathbf{F}({}^{(I_r)}\mathbf{x}, t, {}^{(I_r)}\mathbf{p}) \text{ for all } I_r \in \{1, 2, \cdots, M_r\} \tag{4.111}
$$

There are l-constraints on two subspaces sets $\Omega_{\mathscr{S}}$ and $\Omega_{\mathscr{M}}$,

$$
{}^{(I_r,I_s)}\varphi_j({}^{(I_r,\mu_r)}\mathbf{x},\, {}^{(I_s,\mu_s)}\mathbf{x}, t,\, \boldsymbol{\lambda}_j) = 0 \text{ for all } j \in \{1, 2, \cdots, l\} \tag{4.112}
$$

with $l \leq \Sigma_{I=1}^{M} n_{(I)}$. If all the l-constraints in Eq. (4.112) hold for time $t \in [t_{m_1}, t_{m_2}]$, then M_s-slave systems with M_r-master systems are called to be synchronized for time $t \in [t_{m_1}, t_{m_2}]$ in the sense of Eq. (4.112).

The foregoing definition gives the synchronization between two clusters of slave and master systems are discussed. For $I_r = I_s = 1$, the foregoing definition implies the slave and master systems are one to one. If the two subspace sets of the slave and master systems take all components in state space, and the corresponding constraints in Eq. (4.112) becomes Eq. (3.3) or Eq. (3.4). The synchronicity for such slave and master systems was discussed as before. To further explain the above definition, one slave system with multiple master systems or one master system with multiple slave systems can be discussed.

Definition 4.25 Consider M_s-slave systems from the $\mathscr{S}$-set of slave systems and a master system from the $\mathscr{M}$-set of master systems

$$
{}^{(I_s)}\dot{\mathbf{x}} = {}^{(I_s)}\mathbf{F}({}^{(I_s)}\mathbf{x}, t, {}^{(I_s)}\mathbf{p}) \text{ for all } I_s \in \{1, 2, \cdots, M_s\} \tag{4.113}
$$

$$
{}^{(I_r)}\dot{\mathbf{x}} = {}^{(I_r)}\mathbf{F}({}^{(I_r)}\mathbf{x}, t, {}^{(I_r)}\mathbf{p}) \text{ for } I_r = 1 \tag{4.114}
$$

There are l-constraints on two subspace sets $\Omega_{\mathscr{S}}$ and $\Omega_{\mathscr{M}}$,

$$
{}^{(1,I_s)}\varphi_j({}^{(1)}\mathbf{x},\, {}^{(I_s)}\mathbf{x}, t,\, \boldsymbol{\lambda}_j) = 0 \text{ for all } j \in \{1, 2, \cdots, l\} \tag{4.115}
$$

with $l \leq \Sigma_{I=1}^{M} n_{(I)}$. If all the l-constraints in Eq. (4.115) hold for time $t \in [t_{m_1}, t_{m_2}]$, then the M_s-slave systems with the master system are called to be synchronized for time $t \in [t_{m_1}, t_{m_2}]$ in the sense of Eq. (4.115).

This definition tells that M_s-slave systems are synchronized with one master system with different constraints. For each $I_s \in \{1, 2, \cdots, M_s\}$, the corresponding slave system synchronized with the master system can be discussed. Consider two master systems for M_s-slave systems under different constraints.

Definition 4.26 Consider M_s-slave systems from the $\mathcal{S}$-set of slave systems and two master systems from the $\mathcal{M}$-set of master systems

$$^{(I_s)}\dot{\mathbf{x}} = {^{(I_s)}}\mathbf{F}(^{(I_s)}\mathbf{x}, t, {^{(I_s)}}\mathbf{p}) \text{ for all } I_s \in \{1, 2, \cdots, M_s\} \tag{4.116}$$

$$^{(I_r)}\dot{\mathbf{x}} = {^{(I_r)}}\mathbf{F}(^{(I_r)}\mathbf{x}, t, {^{(I_r)}}\mathbf{p}) \text{ for } I_r = 1, 2 \tag{4.117}$$

There are l-constraints on two subspaces sets $\Omega_{\mathcal{S}}$ and $\Omega_{\mathcal{M}}$,

$$^{(I_r,I_s)}\varphi_j(^{(I_r)}\mathbf{x}, {^{(I_s)}}\mathbf{x}, t, \boldsymbol{\lambda}_j) = 0 \text{ for all } j \in \{1, 2, \cdots, l\} \text{ and } I_r = 1, 2 \tag{4.118}$$

with $l \leq \Sigma_{I=1}^{M} n_{(I)}$. If all the l-constraints in Eq. (4.118) hold for time $t \in [t_{m_1}, t_{m_2}]$, then the M-slave systems with two master systems are called *to be synchronized* for time $t \in [t_{m_1}, t_{m_2}]$ in the sense of Eq. (4.118).

The foregoing definition gives that the M_s-slave systems can be synchronized with two master systems through different constraints. If the two master systems are considered to be two parent systems, the M_s-slave systems are treated as M_s-children systems. Further, the synchronicity of the parent and child systems can be called *the similarity of the parent and child systems*. For each child (or slave) system, under specific constraints in Eq. (4.118), the similarity of the child system with the two parent systems can be investigated as the synchronicity of the slave and master systems as discussed before. The synchronization of a slave system with multiple master systems under certain constraints can be also discussed.

Definition 4.27 Consider a slave system from the $\mathcal{S}$-set of slave systems and M_r-master system from the $\mathcal{M}$-set of master systems

$$^{(I_s)}\dot{\mathbf{x}} = {^{(I_s)}}\mathbf{F}(^{(I_s)}\mathbf{x}, t, {^{(I_s)}}\mathbf{p}) \text{ for } I_s = 1, \tag{4.119}$$

$$^{(I_r)}\dot{\mathbf{x}} = {^{(I_r)}}\mathbf{F}(^{(I_r)}\mathbf{x}, t, {^{(I_r)}}\mathbf{p}) \text{ for all } I_r \in \{1, 2, \cdots, M_r\}. \tag{4.120}$$

There are l-constraints on two subspaces sets $\Omega_{\mathcal{S}}$ and $\Omega_{\mathcal{M}}$,

$$^{(I_r,1)}\varphi_j(^{(I_r)}\mathbf{x}, {^{(1)}}\mathbf{x}, t, \boldsymbol{\lambda}_j) = 0 \text{ for all } j \in \{1, 2, \cdots, l\} \tag{4.121}$$

with $l \leq n_{(1)}$. If all the l-constraints in Eq. (4.121) hold for time $t \in [t_{m_1}, t_{m_2}]$, then the slave system with the M_r-master systems are called to be synchronized for time $t \in [t_{m_1}, t_{m_2}]$ in the sense of Eq. (4.121).

The definition gives the slave system controlled by the M_r-master systems under the l-constraints. Similarly, the synchronization, desynchronization, and penetration on the boundary determined by the constraints can be defined through constraint functions of constraints, and the necessary and sufficient conditions for the synchronicity of complicated synchronizations of multiple systems can be developed in an alike fashion. To extend our discussion, the afore-discussed slave and master systems are considered as two classes of general systems. In other words, the vector fields for two classes of systems can be varied to the corresponding constraints. In addition, it is admissible that any a dynamical system from two classes of dynamical systems can be overconstrained. For this case, Definition 4.27 can be extended and applied for the number of constraints with $1 < l \leq \Sigma_{I_s=1}^{M_s} n_{(I_s)} + \Sigma_{I_r=1}^{M_r} n_{(I_r)}$ as discussed in Chap. 3.

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Chapter 5
Function Synchronizations

In this chapter, the theory of dynamical system synchronization will be applied to the function synchronization of two distinct dynamical systems. Periodic and chaotic synchronizations between two distinct dynamical systems under specific constraints will be investigated. The analytical conditions for the sinusoidal synchronization of the pendulum and Duffing oscillator will be presented, and the invariant domain of sinusoidal synchronization will be discussed. From analytical conditions, the control parameter map will be developed. The function synchronization identification of two distinct dynamical systems with specific constraints must be carried out only by G-functions. The significance of the function synchronization of distinct dynamical systems is to make the synchronicity behaviors hidden, which can be very useful for telecommunication synchronization and network security.

5.1 Synchronization Constraints

In this section, basic concepts of the dynamical system synchronizations will be presented. The discontinuous description of the synchronization of two dynamical systems will be presented.

Definition 5.1 Two dynamical systems are defined by

$$\dot{\mathbf{y}} = \mathbf{F}(\mathbf{y}, t, \mathbf{p}) \in \mathscr{R}^n \text{ and } \dot{\mathbf{x}} = \mathscr{F}(\mathbf{x}, t, \mathbf{q}) \in \mathscr{R}^{\underline{n}} \tag{5.1}$$

If two flows $\mathbf{x}(t)$ and $\mathbf{y}(t)$ of the two systems in Eq. (5.1) satisfy

$$\varphi(\mathbf{x}(t), \mathbf{y}(t), t, \boldsymbol{\lambda}) = 0, \ \boldsymbol{\lambda} \in \mathscr{R}^{n_0}, \tag{5.2}$$

then the two systems are called to be synchronized (or constrained) under such a condition at time t.

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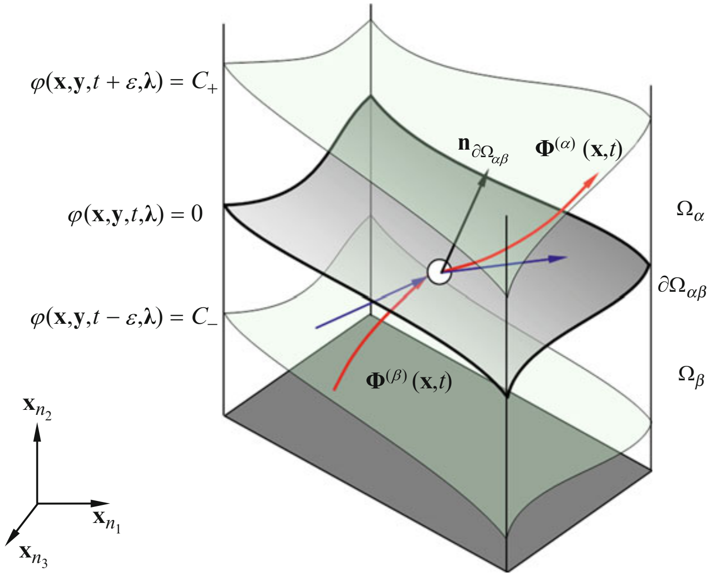

Fig. 5.1 Synchronization surface for the two dynamical systems in Eq. (5.1)

From the foregoing definition, the synchronization (or constraint) of two dynamical systems in Eq. (5.1) occurs through $\varphi(\mathbf{x}(t), \mathbf{y}(t), t, \boldsymbol{\lambda}) = 0$ in Eq. (5.2). Such a condition may cause the discontinuity for two dynamical systems. If the synchronization condition is the separation boundary, then the domain and boundary for the first dynamical system in Eq. (5.1) will be time-varying, which is controlled by a flow of the second dynamical system in Eq. (5.1) (i.e., $\mathbf{x}(t)$), vice versa. Suppose the synchronization of two systems occurs at time t. For time $t \pm \varepsilon$ ($\varepsilon > 0$), there are two constants with

$$\varphi(\mathbf{x}, \mathbf{y}, t \pm \varepsilon, \boldsymbol{\lambda}) = C_\pm \neq 0. \tag{5.3}$$

If the flows of two systems in Eq. (5.1) satisfy Eq. (5.3), then the two systems will not be interacted, as shown in Fig. 5.1. In fact, the synchronization of two dynamical systems can occur under many constraints instead of Eq. (5.2), i.e.,

Definition 5.2 Consider l-non-identical functions of $\varphi_j(\mathbf{x}(t), \mathbf{y}(t), t, \boldsymbol{\lambda}_j)$ ($j \in \mathscr{L}$ and $\mathscr{L} = \{1, 2, \cdots, l\}$). If two flows $\mathbf{x}(t)$ and $\mathbf{y}(t)$ of two systems in Eq. (5.1) satisfy for time t

$$\varphi_j(\mathbf{x}(t), \mathbf{y}(t), t, \boldsymbol{\lambda}_j) = 0 \quad \text{for } \boldsymbol{\lambda}_j \in \mathscr{R}^{n_j} \text{ and } j \in \mathscr{L}, \tag{5.4}$$

then two systems in Eq. (5.1) are called to be synchronized (or constrained) under the jth-condition at time t.

For the foregoing definition, two dynamical systems in Eq. (5.1) possess l-conditions for synchronizations (or constraints). Thus, the l-separation boundaries relative to the synchronization divide the corresponding phase space into many sub-domains for the two dynamical systems, and these sub-domains change with time.

The synchronization between two dynamical systems can be discussed in the vicinity of synchronization boundary. Since the master system is independent of the synchronization constraint, only the slave system should be controlled to satisfy the synchronization constraints. Thus, the controlled slave system is discontinuous under the jth-synchronization constraint. The corresponding domains and boundary for the controlled slave system can be defined by

$$\begin{aligned}\Omega_{(1,j)} &= \left\{ \mathbf{y}^{(1,j)} \middle| \begin{array}{l} \varphi_j(\mathbf{x}(t), \mathbf{y}^{(1,j)}(t), t, \boldsymbol{\lambda}_j) > 0, \\ \varphi_j \text{ is } C^{r_j}\text{-continuous } (r_j \geq 1) \end{array} \right\}, \\ \Omega_{(2,j)} &= \left\{ \mathbf{y}^{(2,j)} \middle| \begin{array}{l} \varphi_j(\mathbf{x}(t), \mathbf{y}^{(2,j)}(t), t, \boldsymbol{\lambda}_j) < 0, \\ \varphi_j \text{ is } C^{r_j}\text{-continuous } (r_j \geq 1) \end{array} \right\},\end{aligned} \tag{5.5}$$

$$\begin{aligned}\partial\Omega_{(12,j)} &= \bar{\Omega}_{(1,j)} \cap \bar{\Omega}_{(2,j)} \\ &= \left\{ \mathbf{y}^{(0,j)} \middle| \begin{array}{l} \varphi_j(\mathbf{x}(t), \mathbf{y}^{(0,j)}(t), t, \boldsymbol{\lambda}_j) = 0 \\ \varphi_j \text{ is } C^{r_j}\text{-continuous } (r_j \geq 1) \end{array} \right\}.\end{aligned} \tag{5.6}$$

From the domains and boundary, the corresponding equations for the controlled slaved system become for $j \in \mathscr{L}$

$$\dot{\mathbf{y}}^{(\alpha_j,j)} = \mathbf{F}^{(\alpha_j,j)}(\mathbf{y}^{(\alpha_j,j)}, t, \mathbf{p}^{(\alpha_j,j)}) \quad \text{on } \Omega_{(\alpha_j,j)} \tag{5.7}$$

$$\dot{\mathbf{y}}^{(0,j)} = \mathbf{F}^{(0,j)}(\mathbf{y}^{(0,j)}, t, \boldsymbol{\lambda}_j) \quad \text{on } \partial\Omega_{(12,j)} \tag{5.8}$$

5.2 Synchronization Mechanism

In this section, the synchronization behaviors between two dynamical systems will be discussed in the vicinity of synchronization boundary. For simplicity, a new variable is introduced in domain $\Omega_{(\alpha_j,j)}$

$$z^{(\alpha_j,j)} = \varphi_j(\mathbf{x}(t), \mathbf{y}^{(\alpha_j,j)}(t), t, \boldsymbol{\lambda}_j) \quad \text{for } j \in \mathscr{L}. \tag{5.9}$$

On the boundary $\partial\Omega_{(\alpha_j\beta_j,j)}$,

$$z^{(0,j)} = \varphi_j(\mathbf{x}(t), \mathbf{y}^{(0,j)}(t), t, \boldsymbol{\lambda}_j) = 0 \quad \text{for } j \in \mathscr{L}. \tag{5.10}$$

If the two systems do not synchronize each other, the new variables ($z_j \neq 0, j = 1, 2, \cdots, l$) will change with time t. The corresponding time-change rate is given by

$$\begin{aligned} \dot{z}^{(\alpha_j, j)} &= D\varphi_j(\mathbf{x}, \mathbf{y}^{(\alpha_j, j)}, t, \boldsymbol{\lambda}_j) = \frac{\partial \varphi_j}{\partial \mathbf{x}} \dot{\mathbf{x}} + \frac{\partial \varphi_j}{\partial \mathbf{y}^{(\alpha_j, j)}} \dot{\mathbf{y}}^{(\alpha_j, j)} + \frac{\partial \varphi_j}{\partial t} \\ &= \sum\nolimits_{p=1}^{\underline{n}} \frac{\partial \varphi_j}{\partial x_p} \dot{x}_p + \sum\nolimits_{q=1}^{n} \frac{\partial \varphi_j}{\partial y_q^{(\alpha_j, j)}} \dot{y}_q^{(\alpha_j, j)} + \frac{\partial \varphi_j}{\partial t}. \end{aligned} \tag{5.11}$$

Substitution of Eqs. (5.1) and (5.7) into Eq. (5.11) yields

$$\begin{aligned} \dot{z}^{(\alpha_j, j)} &= \sum\nolimits_{p=1}^{\underline{n}} \frac{\partial \varphi_j}{\partial x_p} \mathscr{F}_p(\mathbf{x}, t, \mathbf{q}) \\ &\quad + \sum\nolimits_{q=1}^{n} \frac{\partial \varphi_j}{\partial y_q^{(\alpha_j, j)}} F_q^{(\alpha_j, j)}(\mathbf{y}^{(\alpha_j, j)} t, \mathbf{p}^{(\alpha_j, j)}) + \frac{\partial \varphi_j}{\partial t}. \end{aligned} \tag{5.12}$$

Two new normal vectors are defined as

$$\begin{aligned} \underline{\mathbf{n}}_{\varphi_j} &= \frac{\partial \varphi_j}{\partial \mathbf{x}} = \left(\frac{\partial \varphi_j}{\partial x_1}, \frac{\partial \varphi_j}{\partial x_2}, \cdots, \frac{\partial \varphi_j}{\partial x_{\underline{n}}}\right)^{\mathrm{T}}, \\ \mathbf{n}_{\varphi_j} &= \frac{\partial \varphi_j}{\partial \mathbf{y}^{(\alpha_j, j)}} = \left(\frac{\partial \varphi_j}{\partial y_1^{(\alpha_j, j)}}, \frac{\partial \varphi_j}{\partial y_2^{(\alpha_j, j)}}, \cdots, \frac{\partial \varphi_j}{\partial y_n^{(\alpha_j, j)}}\right)^{\mathrm{T}}. \end{aligned} \tag{5.13}$$

Using Eq. (5.13), Eq. (5.12) becomes

$$\dot{z}^{(\alpha_j, j)} = \underline{\mathbf{n}}_{\varphi_j} \cdot \mathscr{F}(\mathbf{x}, t, \mathbf{q}) + \mathbf{n}_{\varphi_j} \cdot \mathbf{F}^{(\alpha_j, j)}(\mathbf{y}^{(\alpha_j, j)}, t, \mathbf{p}^{(\alpha_j, j)}) + \frac{\partial \varphi_j}{\partial t}. \tag{5.14}$$

If the vector fields in different domains $\Omega_{(\alpha_j, j)}$ ($\alpha_j = 1, 2$) are distinguishing, $\dot{z}^{(\alpha_j, j)}$ is discontinuous. Similarly, for each domain $\boldsymbol{\Omega}_{(\alpha_j, j)}$, one obtains

$$\ddot{z}^{(\alpha_j, j)} = D[\underline{\mathbf{n}}_{\varphi_j} \cdot \mathscr{F}(\mathbf{x}, t, \mathbf{q}) + \mathbf{n}_{\varphi_j} \cdot \mathbf{F}^{(\alpha_j, j)}(\mathbf{y}^{(\alpha_j, j)}, t, \mathbf{p}^{(\alpha_j, j)}) + \frac{\partial \varphi_j}{\partial t}]. \tag{5.15}$$

The combination of Eqs. (5.14) and (5.15) gives a dynamical system in phase space of $(z, \dot{z})$, i.e., for $j \in \mathscr{L}$

$$\left.\begin{aligned} \dot{z}^{(\alpha_j, j)} &= g_1^{(\alpha_j, j)}(\mathbf{z}^{(\alpha_j, j)}, t) \\ &\equiv \underline{\mathbf{n}}_{\varphi_j} \cdot \mathscr{F}(\mathbf{x}, t, \mathbf{p}) + \mathbf{n}_{\varphi_j} \cdot \mathbf{F}^{(\alpha_j, j)}(\mathbf{y}^{(\alpha_j, j)}, t, \mathbf{q}^{(\alpha_j, j)}) + \frac{\partial \varphi_j}{\partial t}, \\ \ddot{z}^{(\alpha_j, j)} &= g_2^{(\alpha_j, j)}(\mathbf{z}^{(\alpha_j, j)}, t) \equiv D g_1^{(\alpha_j, j)}(\mathbf{z}^{(\alpha_j, j)}, t) \\ &= D[\underline{\mathbf{n}}_{\varphi_j} \cdot \mathscr{F}(\mathbf{x}, t, \mathbf{p}) + \mathbf{n}_{\varphi_j} \cdot \mathbf{F}^{(\alpha_j, j)}(\mathbf{y}^{(\alpha_j, j)}, t, \mathbf{q}^{(\alpha_j, j)}) + \frac{\partial \varphi_j}{\partial t}], \end{aligned}\right\} \tag{5.16}$$

where $\mathbf{z}^{(\alpha_j,j)} = (z^{(\alpha_j,j)}, \dot{z}^{(\alpha_j,j)})^{\mathrm{T}}$. Letting $\mathbf{g}^{(\alpha_j,j)} = (g_1^{(\alpha_j,j)}, g_2^{(\alpha_j,j)})^{\mathrm{T}}$, one obtains

$$\left.\begin{aligned} \dot{\mathbf{z}}^{(\alpha_j,j)} &= \mathbf{g}^{(\alpha_j,j)}(\mathbf{z}^{(\alpha_j,j)}, t) \text{ for } j \in \mathscr{L}; \\ \dot{\mathbf{x}} &= \mathscr{F}(\mathbf{x}, t, \mathbf{q}) \in \mathscr{R}^{\underline{n}}, \\ \dot{\mathbf{y}}^{(\alpha_j,j)} &= \mathbf{F}^{(\alpha_j,j)}(\mathbf{y}^{(\alpha_j,j)}, t, \mathbf{p}^{(\alpha_j,j)}) \in \mathscr{R}^n. \end{aligned}\right\} \tag{5.17}$$

For a better understanding of such a discontinuous dynamical system, the boundary and domains in phase space are defined as

$$\begin{aligned} \partial\Xi_{(12,j)} &= \overline{\Xi}_{(1,j)} \cap \overline{\Xi}_{(2,j)} \\ &= \left\{ (z^{(0,j)}, \dot{z}^{(0,j)}) \middle| \Psi_j(z^{(0,j)}, \dot{z}^{(0,j)}) = z^{(0,j)} = 0 \right\} \subset \mathscr{R}; \end{aligned} \tag{5.18}$$

and

$$\begin{aligned} \Xi_{(1,j)} &= \left\{ (z^{(1,j)}, \dot{z}^{(1,j)}) \middle| z^{(1,j)} > 0 \right\} \subset \mathscr{R}^2; \\ \Xi_{(2,j)} &= \left\{ (z^{(1,j)}, \dot{z}^{(1,j)}) \middle| z^{(2,j)} < 0 \right\} \subset \mathscr{R}^2. \end{aligned} \tag{5.19}$$

$\varphi_j(\mathbf{x}(t), \mathbf{y}^{(0,j)}(t), t, \boldsymbol{\lambda}_j) = 0$ on the synchronization boundary gives

$$\frac{d^s z^{(0,j)}}{dt^s} = D^s \varphi_j(\mathbf{x}(t), \mathbf{y}^{(0,j)}(t), t, \boldsymbol{\lambda}_j) = 0 \text{ for } s = 1, 2, \cdots \tag{5.20}$$

Thus, the synchronization boundary is determined by

$$\begin{aligned} & z^{(0,j)} = 0, \; \dot{z}^{(0,j)} = 0 \quad \text{for } j \in \mathscr{L}; \\ & \dot{\mathbf{x}} = \mathscr{F}(\mathbf{x}, t, \mathbf{q}) \in \mathscr{R}^{\underline{n}}, \text{ and } \dot{\mathbf{y}}^{(0,j)} = \mathbf{F}^{(0,j)}(\mathbf{y}^{(0,j)}, t, \boldsymbol{\lambda}_j) \in \mathscr{R}^n. \end{aligned} \tag{5.21}$$

The domains and boundary in phase space of $(z^{(j)}, \dot{z}^{(j)})$ are sketched in Fig. 5.2 and the location for switching may not be continuous (i.e., $\mathbf{z}^{(\alpha_j,j)} \neq \mathbf{z}^{(\beta_j,j)} \neq \mathbf{z}^{(0,j)} = 0$) because the vector fields of the resultant system are discontinuous (or $\dot{z}^{(\alpha_j,j)} \neq \dot{z}^{(\beta_j,j)} \neq \dot{z}^{(0,j)} = 0$), but the boundary in such phase space is independent of time. However, the boundaries and domains in phase space of the controlled slave system in Eqs. (5.8) and (5.9) are shown in Fig. 5.2. The boundary varying with time is presented, but switching points for a flow are continuous. However, such flows will be controlled by the vector fields $\mathbf{g}^{(1,j)}(\mathbf{z}^{(1,j)}, t)$ and $\mathbf{g}^{(2,j)}(\mathbf{z}^{(2,j)}, t)$. The dynamical systems in phase space $(z^{(j)}, \dot{z}^{(j)})$ are:

$$\left.\begin{aligned} & \dot{\mathbf{z}}^{(\Lambda_j,j)} = \mathbf{g}^{(\Lambda_j,j)}(\mathbf{z}^{(\Lambda_j,j)}, t) \text{ for } j \in \mathscr{L}, \; \Lambda_j = 0, \alpha_j \\ & \dot{\mathbf{x}} = \mathscr{F}(\mathbf{x}, t, \mathbf{q}) \in \mathscr{R}^{\underline{n}}, \\ & \dot{\mathbf{y}}^{(\Lambda_j,j)} = \mathbf{F}^{(\Lambda_j,j)}(\mathbf{y}^{(\Lambda_j,j)}, t, \mathbf{p}^{(\Lambda_j,j)}) \in \mathscr{R}^n, \end{aligned}\right\} \tag{5.22}$$

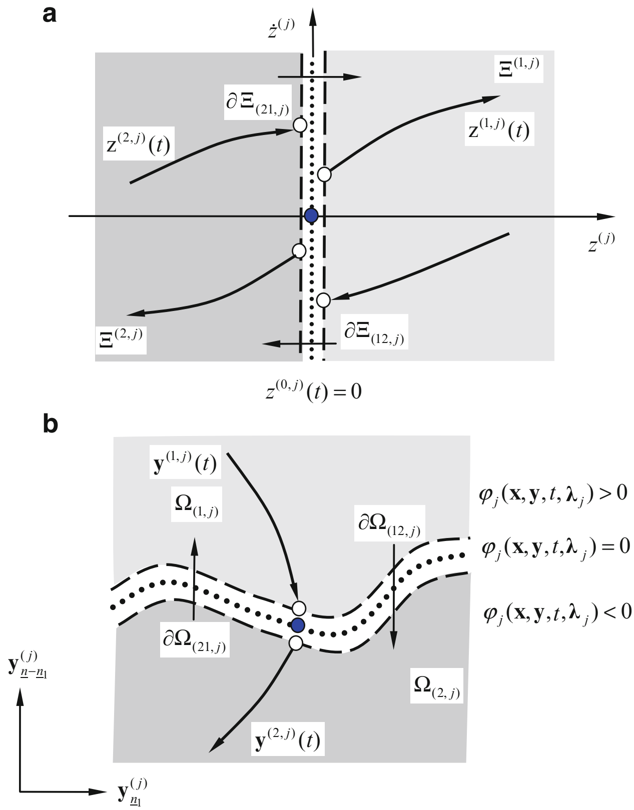

Fig. 5.2 A partition of phase space: (**a**) $(z,\dot{z})$ for the *j*th-synchronization boundary and (**b**) the controlled slave system. Two *dashed lines* (curves) are infinitesimally close to the boundary with the *dotted line* (curve)

where

$$\left.\begin{array}{l}\mathbf{g}^{(\alpha_j,j)}(\mathbf{z}^{(\alpha_j,j)},t)=(g_1^{(\alpha_j,j)}(\mathbf{z}^{(\alpha_j,j)},t),g_2^{(\alpha_j,j)}(\mathbf{z}^{(\alpha_j,j)},t))^{\mathrm{T}}\\ \qquad \text{in } \Xi_{\alpha_j}(\alpha_j\in\{1,2\});\\ \mathbf{g}^{(0,j)}(\mathbf{z}^{(\alpha_j,j)},t)\in[\mathbf{g}^{(\alpha_j,j)}(\mathbf{z}^{(\alpha_j,j)},t),\mathbf{g}^{(\beta_j,j)}(\mathbf{z}^{(\beta_j,j)},t)]\\ \qquad \text{on } \partial\Xi_{(12,j)} \text{ for non-stick,}\\ \mathbf{g}^{(0,j)}(\mathbf{z}^{(\alpha_j,j)},t)=(0,0)^{T}\text{on } \partial\Xi_{(12,j)} \text{ for stick.}\end{array}\right\} \tag{5.23}$$

The normal vector of $\partial\Xi_{(12,j)}$ is computed from Eq. (5.17), i.e.,

$$\mathbf{n}_{\partial\Xi_{(12,j)}}=(1,0)^{\mathrm{T}} \text{ and } D\mathbf{n}_{\partial\Xi_{(12,j)}}=(0,0)^{\mathrm{T}}, \tag{5.24}$$

where $D(\cdot) = D(\cdot)/Dt$. From Chap. 2, the corresponding two G-functions are computed by

$$\begin{aligned} G^{(0,\alpha_j)}_{\partial\Xi_{(12,j)}}(\mathbf{z}^{(\alpha_j,j)},t) &= \mathbf{n}_{\partial\Xi_{(12,j)}} \cdot \mathbf{g}^{(\alpha_j,j)}(\mathbf{z}^{(\alpha_j,j)},t) = g_1^{(\alpha_j,j)}(\mathbf{z}^{(\alpha_j,j)},t), \\ G^{(1,\alpha_j)}_{\partial\Xi_{(12,j)}}(\mathbf{z}^{(\alpha_j,j)},t) &= \mathbf{n}_{\partial\Xi_{(12,j)}} \cdot D\mathbf{g}^{(\alpha_j,j)}(\mathbf{z}^{(\alpha_j,j)},t) = g_2^{(\alpha_j,j)}(\mathbf{z}^{(\alpha_j,j)},t). \end{aligned} \tag{5.25}$$

With G-functions, the sufficient and necessary conditions for a passable flow at $(\mathbf{z}_m^{(0,j)}, t_m)$ with $\mathbf{z}_m^{(\alpha_j,j)} = \mathbf{z}_m^{(0,j)}$ for the boundary $\partial\Xi_{(12,j)}$ are given from Chap. 2

$$\begin{aligned} &\left.\begin{aligned} G^{(0,1)}_{\partial\Xi_{(12,j)}}(\mathbf{z}_m^{(1,j)},t_{m-}) &= g_1^{(1,j)}(\mathbf{z}_m^{(1,j)},t_{m-})<0, \\ G^{(0,2)}_{\partial\Xi_{(12,j)}}(\mathbf{z}_m^{(2,j)},t_{m+}) &= g_1^{(2,j)}(\mathbf{z}_m^{(2,j)},t_{m+})<0, \end{aligned}\right\} \quad \text{for } \Xi_{(1,j)} \to \Xi_{(2,j)} \\ &\left.\begin{aligned} G^{(0,1)}_{\partial\Xi_{(12,j)}}(\mathbf{z}_m^{(1,j)},t_{m+}) &= g_1^{(1,j)}(\mathbf{z}_m^{(1,j)},t_{m+})>0, \\ G^{(0,2)}_{\partial\Xi_{(12,j)}}(\mathbf{z}_m^{(2,j)},t_{m-}) &= g_1^{(2,j)}(\mathbf{z}_m^{(2,j)},t_{m-})>0, \end{aligned}\right\} \quad \text{for } \Xi_{(2,j)} \to \Xi_{(1,j)} \end{aligned} \tag{5.26}$$

where

$$g_1^{(\alpha_j,j)}(\mathbf{z}_m^{(\alpha_j,j)},t_{m\pm}) = \underline{\mathbf{n}}_{\varphi_j} \cdot \mathscr{F}(\mathbf{x}_m,t_{m\pm},\mathbf{q}) + \mathbf{n}_{\varphi_j} \cdot \mathbf{F}^{(\alpha_j,j)}(\mathbf{y}_m^{(\alpha_j,j)},t_{m\pm},\mathbf{p}^{(\alpha_j,j)}) + \frac{\partial\varphi_j}{\partial t}. \tag{5.27}$$

The foregoing conditions give the sufficient and necessary conditions for the controlled slave system synchronizing with the master system under the jth-synchronization condition, and the current states of the controlled slave system will be switched from one domain to the other through such a synchronization condition. From Chap. 3, such a flow to the synchronization boundary is called a *penetration synchronization* between two dynamical systems.

The sufficient and necessary conditions for a stick flow (or sink flow or sliding flow) on the synchronization boundary $\partial\Xi_{(12,j)}$ are obtained from Chap. 2,

$$\left.\begin{aligned} G^{(0,1)}_{\partial\Xi_{(12,j)}}(\mathbf{z}_m^{(1,j)},t_{m-}) &= g_1^{(1,j)}(\mathbf{z}_m^{(1,j)},t_{m-})<0, \\ G^{(0,2)}_{\partial\Xi_{(12,j)}}(\mathbf{z}_m^{(2,j)},t_{m-}) &= g_1^{(2,j)}(\mathbf{z}_m^{(2,j)},t_{m-})>0 \end{aligned}\right\} \quad \text{on } \partial\Xi_{(12,j)} \tag{5.28}$$

From the foregoing condition, the controlled slave systems will stick with the master system under the jth-synchronization condition. From Chap. 3, this phenomenon is called the synchronization of the controlled slave system with the master system under the jth-synchronization condition.

Similarly, the sufficient and necessary conditions for a source flow on the boundary $\partial\Xi_{(12,j)}$ are given in Chaps. 2–4, i.e.,

$$\left.\begin{aligned} G^{(0,1)}_{\partial\Xi_{(12,j)}}(\mathbf{z}^{(1,j)}_m, t_{m+}) = g^{(1,j)}_1(\mathbf{z}^{(1,j)}_m, t_{m+}) > 0,\\ G^{(0,2)}_{\partial\Xi_{(12,j)}}(\mathbf{z}^{(2,j)}_m, t_{m+}) = g^{(2,j)}_1(\mathbf{z}^{(2,j)}_m, t_{m+}) < 0 \end{aligned}\right\} \text{ on } \partial\Xi_{(12,j)} \tag{5.29}$$

For this case, the controlled slave system will not synchronize with the master system at $(\mathbf{z}^{(0,j)}_m, t_m)$ for the synchronization boundary $\partial\Xi_{(12,j)}$ relative to the jth-synchronization condition. From Chap. 3, this phenomenon is called the desynchronization of the controlled slave system with the master system under the jth-synchronization condition.

The appearance and disappearance of three synchronization states of the two dynamical systems to the jth-synchronization condition in Eq. (5.4) can be determined from Chaps. 2 to 4 (e.g., [1, 2]) for the switching bifurcation of three states of synchronizations between the two dynamical systems.

(i) The sufficient and necessary conditions of synchronization appearance from the instantaneous penetration synchronization are

$$\begin{aligned} &(-1)^{\alpha_j} G^{(0,\alpha_j)}_{\partial\Xi_{(12,j)}}(\mathbf{z}^{(\alpha_j,j)}_m, t_{m-}) = (-1)^{\alpha_j} g^{(\alpha_j,j)}_1(\mathbf{z}^{(\alpha_j,j)}_m, t_{m-}) > 0;\\ &G^{(0,\beta_j)}_{\partial\Xi_{(12,j)}}(\mathbf{z}^{(\beta_j,j)}_m, t_{m\pm}) = g^{(\beta_j,j)}_1(\mathbf{z}^{(\beta_j,j)}_m, t_{m\pm}) = 0,\\ &(-1)^{\beta_j} G^{(1,\beta_j)}_{\partial\Xi_{(12,j)}}(\mathbf{z}^{(\beta_j,j)}_m, t_{m\pm}) = (-1)^{\beta_j} g^{(\beta_j,j)}_2(\mathbf{z}^{(\beta_j,j)}_m, t_{m\pm}) < 0. \end{aligned} \tag{5.30}$$

The sufficient and necessary conditions for synchronization vanishing from the jth-synchronization boundary are

$$\begin{aligned} &(-1)^{\alpha_j} G^{(0,\alpha_j)}_{\partial\Xi_{(\alpha_i\beta_j,j)}}(\mathbf{z}^{(\alpha_j,j)}_m, t_{m-}) = (-1)^{\alpha_j} g^{(\alpha_j,j)}_1(\mathbf{z}^{(\alpha_j,j)}_m, t_{m-}) > 0;\\ &G^{(0,\beta_j)}_{\partial\Xi_{(\alpha_i\beta_j,j)}}(\mathbf{z}^{(\beta_j,j)}_m, t_{m\mp}) = g^{(\beta_j,j)}_1(\mathbf{z}^{(\beta_j,j)}_m, t_{m\mp}) = 0,\\ &(-1)^{\beta_j} G^{(1,\beta_j)}_{\partial\Xi_{(\alpha_i\beta_j,j)}}(\mathbf{z}^{(\beta_j,j)}_m, t_{m\mp}) = (-1)^{\beta_j} g^{(\beta_j,j)}_2(\mathbf{z}^{(\beta_j,j)}_m, t_{m\mp}) < 0. \end{aligned} \tag{5.31}$$

The *appearance* and *vanishing* conditions for the synchronization relative to the instantaneous synchronization in Eq. (5.30) are the *vanishing* and *appearance* conditions for the instantaneous penetration synchronization relative to the synchronization, respectively.

(ii) From Chaps. 2 to 4, the sufficient and necessary conditions are

$$\begin{aligned} &(-1)^{\alpha_j} G^{(0,\alpha_j)}_{\partial\Xi_{(\alpha_i\beta_j,j)}}(\mathbf{z}^{(\alpha_j,j)}_m, t_{m+}) = (-1)^{\alpha_j} g^{(\alpha_j,j)}_1(\mathbf{z}^{(\alpha_j,j)}_m, t_{m+}) < 0;\\ &G^{(0,\beta_j)}_{\partial\Xi_{(\alpha_i\beta_j,j)}}(\mathbf{z}^{(\beta_j,j)}_m, t_{m\mp}) = g^{(\beta_j,j)}_1(\mathbf{z}^{(\beta_j,j)}_m, t_{m\mp}) = 0,\\ &(-1)^{\beta_j} G^{(1,\beta_j)}_{\partial\Xi_{(\alpha_i\beta_j,j)}}(\mathbf{z}^{(\beta_j,j)}_m, t_{m\mp}) = (-1)^{\beta_j} g^{(\beta_j,j)}_2(\mathbf{z}^{(\beta_j,j)}_m, t_{m\mp}) < 0 \end{aligned} \tag{5.32}$$

for desynchronization appearance pertaining to the instantaneous penetration synchronization and,

$$\begin{aligned}
&(-1)^{\alpha_j} G^{(0,\alpha_j)}_{\partial\Xi_{(\alpha_i\beta_j,j)}}(\mathbf{z}_m^{(\alpha_j,j)}, t_{m+}) = (-1)^{\alpha_j} g_1^{(\alpha_j,j)}(\mathbf{z}_m^{(\alpha_j,j)}, t_{m+}) < 0;\\
&G^{(0,\beta_j)}_{\partial\Xi_{(\alpha_i\beta_j,j)}}(\mathbf{z}_m^{(\beta_j,j)}, t_{m\pm}) = g_1^{(\beta_j,j)}(\mathbf{z}_m^{(\beta_j,j)}, t_{m\pm}) = 0,\\
&(-1)^{\beta_j} G^{(1,\beta_j)}_{\partial\Xi_{(\alpha_i\beta_j,j)}}(\mathbf{z}_m^{(\beta_j,j)}, t_{m\pm}) = (-1)^{\beta_j} g_2^{(\beta_j,j)}(\mathbf{z}_m^{(\beta_j,j)}, t_{m\pm}) < 0.
\end{aligned} \tag{5.33}$$

for the vanishing of the desynchronization pertaining to the instantaneous synchronization.

(iii) From Chaps. 2 to 4, the sufficient and necessary switching conditions between the synchronization and desynchronization of the controlled slave and master systems on the jth-synchronization boundary are

$$\begin{aligned}
&G^{(0,\alpha_j)}_{\partial\Xi_{(\alpha_i\beta_j,j)}}(\mathbf{z}_m^{(\alpha_j,j)}, t_{m\mp}) = g_1^{(\alpha_j,j)}(\mathbf{z}_m^{(\alpha_j,j)}, t_{m\mp}) = 0\\
&(-1)^{\alpha_j} G^{(1,\alpha_j)}_{\partial\Xi_{(\alpha_i\beta_j,j)}}(\mathbf{z}_m^{(\alpha_j,j)}, t_{m\mp}) = (-1)^{\alpha_j} g_2^{(\alpha_j,j)}(\mathbf{z}_m^{(\alpha_j,j)}, t_{m\mp}) < 0;\\
&G^{(0,\beta_j)}_{\partial\Xi_{(\alpha_i\beta_j,j)}}(\mathbf{z}_m^{(\beta_j,j)}, t_{m\mp}) = g_1^{(\beta_j,j)}(\mathbf{z}_m^{(\beta_j,j)}, t_{m\mp}) = 0,\\
&(-1)^{\beta_j} G^{(1,\beta_j)}_{\partial\Xi_{(\alpha_i\beta_j,j)}}(\mathbf{z}_m^{(\beta_j,j)}, t_{m\mp}) = (-1)^{\beta_j} g_2^{(\beta_j,j)}(\mathbf{z}_m^{(\beta_j,j)}, t_{m\mp}) < 0.
\end{aligned} \tag{5.34}$$

Similarly, the sufficient and necessary switching conditions between two instantaneous penetration synchronizations at the jth-synchronization boundary for $\alpha_j \neq \beta_j$ are

$$\begin{aligned}
&G^{(0,\alpha_j)}_{\partial\Xi_{(12,j)}}(\mathbf{z}_m^{(\alpha_j,j)}, t_{m\mp}) = g_1^{(\alpha_j,j)}(\mathbf{z}_m^{(\alpha_j,j)}, t_{m\mp}) = 0 \quad \text{for } \alpha_j \in \{1,2\},\\
&(-1)^{\alpha_j} G^{(1,\alpha_j)}_{\partial\Xi_{(12,j)}}(\mathbf{z}_m^{(\alpha_j,j)}, t_{m\mp}) = (-1)^{\alpha_j} g_2^{(\alpha_j,j)}(\mathbf{z}_m^{(\alpha_j,j)}, t_{m\mp}) < 0;\\
&G^{(0,\beta_j)}_{\partial\Xi_{(12,j)}}(\mathbf{z}_m^{(\beta_j,j)}, t_{m\pm}) = g_1^{(\beta_j,j)}(\mathbf{z}_m^{(\beta_j,j)}, t_{m\pm}) = 0 \quad \text{for } \beta_j \in \{1,2\},\\
&(-1)^{\beta_j} G^{(1,\beta_j)}_{\partial\Xi_{(12,j)}}(\mathbf{z}_m^{(\beta_j,j)}, t_{m\pm}) = (-1)^{\beta_j} g_2^{(\beta_j,j)}(\mathbf{z}_m^{(\beta_j,j)}, t_{m\pm}) < 0.
\end{aligned} \tag{5.35}$$

A flow of the controlled slave system, tangential to the synchronization boundary $\partial\Xi_{(12,j)}$ is another instantaneous synchronization (or tangential synchronization), and the corresponding sufficient and necessary conditions are

$$\begin{aligned}
&G^{(0,\alpha_j)}_{\partial\Xi_{(12,j)}}(\mathbf{z}_m^{(\alpha_j,j)}, t_{m\pm}) = g_1^{(\alpha_j,j)}(\mathbf{z}_m^{(\alpha_j,j)}, t_{m\pm}) = 0 \quad \text{for } \alpha_j \in \{1,2\},\\
&(-1)^{\alpha_j} G^{(1,\alpha_j)}_{\partial\Xi_{(12,j)}}(\mathbf{z}_m^{(\alpha_j,j)}, t_{m\pm}) = (-1)^{\alpha_j} g_2^{(\alpha_j,j)}(\mathbf{z}_m^{(\alpha_j,j)}, t_{m\pm}) < 0.
\end{aligned} \tag{5.36}$$

5.3 Sinusoidal Synchronization

To demonstrate the function synchronization under specific constraints, the periodically forced, damped Duffing oscillator is considered herein as a master system as in Luo [3], i.e.,

$$\ddot{x} + d\dot{x} - a_1 x + a_2 x^3 = A_0 \cos \omega t, \tag{5.37}$$

where x is the displacement, d is the damping coefficient, and A_0 and ω are the excitation amplitude and frequency for the Duffing oscillator. A parametrically excited chaotic pendulum is considered as a slave system in Luo and Han [4], i.e.,

$$\ddot{y} + a_0 \sin y = Q_0 \cos \Omega t, \tag{5.38}$$

where y is the displacement for pendulum, and Q_0 and Ω are excitation amplitude and frequency, respectively. Consider a sinusoidal constraint of two displacements as

$$\varphi_1 = y - \sin x = 0. \tag{5.39}$$

Due to the velocity $\dot{x} = dx/dt$ and $\dot{y} = dy/dt$, the velocity synchronization constraint for dynamical systems is given by

$$\varphi_2 = \dot{y} - \dot{x} \cos x = 0. \tag{5.40}$$

The state variables for the master and slave systems are

$$\mathbf{x} = (x_1, x_2)^{\mathrm{T}} \equiv (x, \dot{x})^{\mathrm{T}} \text{ and } \mathbf{y} = (y_1, y_2)^{\mathrm{T}} \equiv (y, \dot{y})^{\mathrm{T}}. \tag{5.41}$$

and the vector fields for master and slave systems are

$$\mathscr{F}(\mathbf{x}, t) = (\mathscr{F}_1(\mathbf{x}, t), \mathscr{F}_2(\mathbf{x}, t))^{\mathrm{T}} \text{ and } \overline{\mathbf{F}}(\mathbf{y}, t) = (\overline{F}_1(\mathbf{y}, t), \overline{F}_2(\mathbf{y}, t))^{\mathrm{T}}, \tag{5.42}$$

where

$$x_2 = \dot{x}_1 \text{ and } \mathscr{F}(\mathbf{x}, t) = -d_1 x_2 + a_1 x_1 - a_2 x_1^3 + A_0 \cos \omega t. \tag{5.43}$$

$$\bar{F}_1(\mathbf{y}, t) = y_2 \text{ and } \bar{F}_2(\mathbf{y}, t) = -a_0 \sin y_1 + Q_0 \cos \Omega t. \tag{5.44}$$

To make the chaotic pendulum system synchronizing with the Duffing oscillator with the sinusoidal constraints in Eqs. (5.39) and (5.40), a feedback controller is designed as

$$\begin{aligned} &\mathbf{u}(\mathbf{x}, \mathbf{y}, t, \mathbf{k}) = (u_1, u_2)^{\mathrm{T}} \\ &u_1 = -k_1 \operatorname{sgn}(y_1 - \sin x_1) \text{ and } u_2 = -k_2 \operatorname{sgn}(y_2 - x_2 \cos x_1), \end{aligned} \tag{5.45}$$

where k_1 and k_2 are controller parameters. Thus the controlled pendulum becomes

$$\dot{\mathbf{y}} = \mathbf{F}(\mathbf{y},t) \equiv \overline{\mathbf{F}}(\mathbf{y},t) + \mathbf{u}(\mathbf{x},\mathbf{y},t), \tag{5.46}$$

where the vector fields are defined by $\mathbf{F}(\mathbf{y},t) = (F_1(\mathbf{y},t), F_2(\mathbf{y},t))^{\mathrm{T}}$. With the control laws, there are four regions

(i) For $y_1 > \sin x_1$ and $y_2 > x_2 \cos x_1$,

$$\begin{aligned} F_1(\mathbf{y},t) &= y_2 - k_1, \\ F_2(\mathbf{y},t) &= -a_0 \sin y_1 + Q_0 \cos \Omega t - k_2. \end{aligned} \tag{5.47}$$

(ii) For $y_1 > \sin x_1$ and $y_2 < x_2 \cos x_1$,

$$\begin{aligned} F_1(\mathbf{y},t) &= y_2 - k_1, \\ F_2(\mathbf{y},t) &= -a_0 \sin y_1 + Q_0 \cos \Omega t + k_2. \end{aligned} \tag{5.48}$$

(iii) For $y_1 < \sin x_1$ and $y_2 < x_2 \cos x_1$,

$$\begin{aligned} F_1(\mathbf{y},t) &= y_2 - k_1, \\ F_2(\mathbf{y},t) &= -a_0 \sin y_1 + Q_0 \cos \Omega t + k_2. \end{aligned} \tag{5.49}$$

(iv) For $y_1 < \sin x_1$ and $y_2 > x_2 \cos x_1$,

$$\begin{aligned} F_1(\mathbf{y},t) &= y_2 + k_1, \\ F_2(\mathbf{y},t) &= -a_0 \sin y_1 + Q_0 \cos \Omega t - k_2. \end{aligned} \tag{5.50}$$

Under the control laws, the controlled pendulum system has four regions, four boundaries, plus an intersection point with different dynamical systems. The intersection point is the synchronization of the controlled pendulum synchronizing with the Duffing oscillator. In phase space, four domains Ω_α $(\alpha = 1,2,3,4)$ of the controlled pendulum are

$$\begin{aligned} \Omega_1 &= \{(y_1,y_2) | y_1 - \sin x_1 > 0, y_2 - x_2 \cos x_1 > 0\}, \\ \Omega_2 &= \{(y_1,y_2) | y_1 - \sin x_1 > 0, y_2 - x_2 \cos x_1 < 0\}, \\ \Omega_3 &= \{(y_1,y_2) | y_1 - \sin x_1 < 0, y_2 - x_2 \cos x_1 < 0\}, \\ \Omega_4 &= \{(y_1,y_2) | y_1 - \sin x_1 < 0, y_2 - x_2 \cos x_1 > 0\}. \end{aligned} \tag{5.51}$$

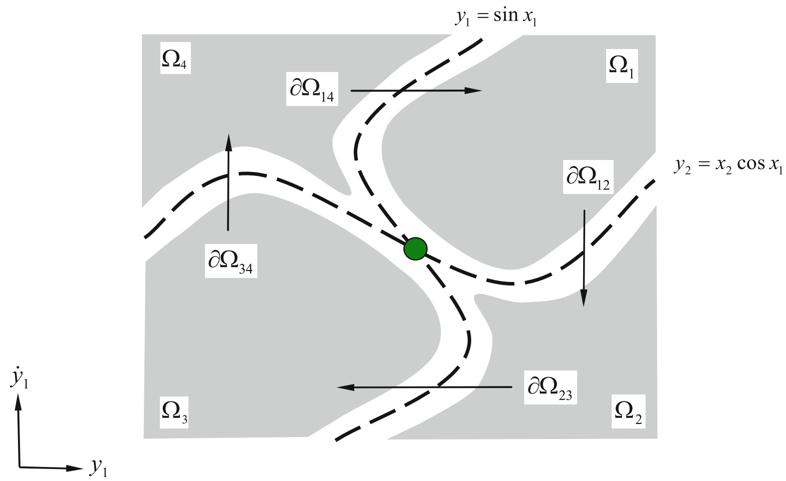

Fig. 5.3 Phase plane partitions and boundaries in the absolute frame

The boundary $\partial\Omega_{\alpha\beta}$ $(\alpha, \beta = 1, 2, 3, 4; \alpha \neq \beta)$ of the four domains are

$$\begin{aligned}
\partial\Omega_{12} &= \{(y_1, y_2) | y_2 - x_2 \cos x_1 = 0, y_1 - \sin x_1 > 0\}, \\
\partial\Omega_{23} &= \{(y_1, y_2) | y_1 - \sin x_1 = 0, y_2 - x_2 \cos x_1 < 0\}, \\
\partial\Omega_{34} &= \{(y_1, y_2) | y_2 - x_2 \cos x_1 = 0, y_1 - \sin x_1 < 0\}, \\
\partial\Omega_{14} &= \{(y_1, y_2) | y_1 - \sin x_1 = 0, y_2 - x_2 \cos x_1 > 0\}.
\end{aligned} \tag{5.52}$$

The intersection point (vertex) of the boundary $\partial\Omega_{\alpha\beta}$ $(\alpha, \beta = 1, 2, 3, 4; \alpha \neq \beta)$ is

$$\begin{aligned}
\angle\partial\Omega_{\alpha\beta} &= \cap_{\alpha=1}^{4} \cap_{\beta=1}^{4} \partial\Omega_{\alpha\beta} \\
&= \{(y_1, y_2) | y_2 - x_2 \cos x_1 = 0, y_1 - \sin x_1 = 0\}.
\end{aligned} \tag{5.53}$$

The domains, boundaries, and vertex for the controlled pendulum are sketched in Fig. 5.3 for dynamical system synchronization under specific constraints through the theory of discontinuous dynamical systems. The domains of the controlled slave system are shaded, and the corresponding displacement and velocity boundaries are dashed curves, controlled by the master system (i.e., the Duffing oscillator). The intersected point of two boundaries is the corner for synchronization, which is labeled by a filled circular symbol.

The dynamical system of the controlled pendulum in the α-domain is

$$\dot{\mathbf{y}}^{(\alpha)} = \mathbf{F}^{(\alpha)}(\mathbf{y}^{(\alpha)}, t), \tag{5.54}$$

where

$$
\begin{aligned}
&\mathbf{F}^{(\alpha)}(\mathbf{y}^{(\alpha)},t) = (F_1^{(\alpha)},F_2^{(\alpha)})^{\mathrm{T}},\\
&F_1^{(\alpha)}(\mathbf{y}^{(\alpha)},t) = y_2^{(\alpha)} - k_1 \text{ for } \alpha = 1,2,\\
&F_1^{(\alpha)}(\mathbf{y}^{(\alpha)},t) = y_2^{(\alpha)} + k_1 \text{ for } \alpha = 3,4,\\
&F_2^{(\alpha)}(\mathbf{y}^{(\alpha)},t) = -a_0 \sin y_1^{(\alpha)} + Q_0 \cos \Omega t - k_2 \text{ for } \alpha = 1,4,\\
&F_2^{(\alpha)}(\mathbf{y}^{(\alpha)},t) = -a_0 \sin y_1^{(\alpha)} + Q_0 \cos \Omega t + k_2 \text{ for } \alpha = 2,3.
\end{aligned}
\tag{5.55}
$$

The dynamical system on boundary $\partial\Omega_{\alpha\beta}$ is

$$
\begin{aligned}
&\dot{\mathbf{y}}^{(\alpha\beta)} = \mathbf{F}^{(\alpha\beta)}(\mathbf{y}^{(\alpha\beta)},\mathbf{x}(t),t),\\
&\dot{\mathbf{x}} = \mathscr{F}(\mathbf{x},t),
\end{aligned}
\tag{5.56}
$$

where

$$
\begin{aligned}
&\mathbf{F}^{(\alpha\beta)} = (F_1^{(\alpha\beta)},F_2^{(\alpha\beta)})^{\mathrm{T}},\\
&F_1^{(\alpha\beta)}(\mathbf{y}^{(\alpha\beta)},t) = y_2^{(\alpha\beta)} = x_2 \cos x_1 \text{ and } F_2^{(\alpha\beta)}(\mathbf{y}^{(\alpha\beta)},t) = \dot{y}_2^{(\alpha\beta)}
\end{aligned}
\tag{5.57}
$$

with

$$
\begin{aligned}
&y_1^{(\alpha\beta)} = \sin x_1 \text{ and } y_2^{(\alpha\beta)} = x_2 \cos x_1 \text{ on } \partial\Omega_{\alpha\beta} \text{ for } (\alpha,\beta) = (2,3),(1,4);\\
&y_1^{(\alpha\beta)} = \sin x_1 + C \text{ and } y_2^{(\alpha\beta)} = x_2 \cos x_1 \text{ on } \partial\Omega_{\alpha\beta} \text{ for } (\alpha,\beta) = (1,2),(3,4).
\end{aligned}
\tag{5.58}
$$

Based on the absolute coordinates, the boundaries, and corner of the controlled pendulum are dependent on time. However, it is very difficult to develop analytical conditions for synchronization. Without loss of generality, the relative coordinates are defined as

$$
z_1 = y_1 - \sin x_1 \text{ and } \dot{z}_1 \equiv z_2 = y_2 - x_2 \cos x_1. \tag{5.59}
$$

The domains, displacement and velocity boundaries, and synchronization corner in the relative coordinates are expressed by

$$
\begin{aligned}
&\Omega_1 = \{(z_1,z_2)\big| z_1 > 0, z_2 > 0\},\\
&\Omega_2 = \{(z_1,z_2)\big| z_1 > 0, z_2 < 0\},\\
&\Omega_3 = \{(z_1,z_2)\big| z_1 < 0, z_2 < 0\},\\
&\Omega_4 = \{(z_1,z_2)\big| z_1 < 0, z_2 > 0\}.
\end{aligned}
\tag{5.60}
$$

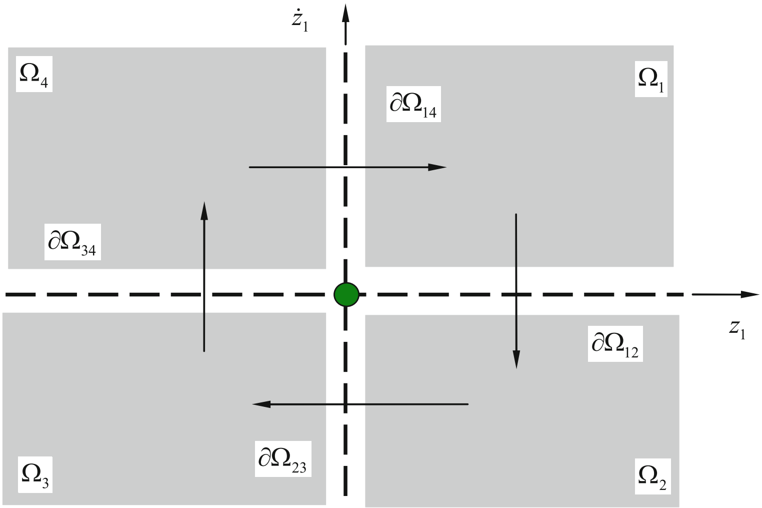

Fig. 5.4 Phase plane partitions and boundaries in the relative frame

$$\begin{aligned}
\partial\Omega_{12} &= \{(z_1, z_2) | z_2 = 0, z_1 > 0\},\\
\partial\Omega_{23} &= \{(z_1, z_2) | z_1 = 0, z_2 < 0\},\\
\partial\Omega_{34} &= \{(z_1, z_2) | z_2 = 0, z_1 < 0\},\\
\partial\Omega_{14} &= \{(z_1, z_2) | z_1 = 0, z_2 > 0\}.
\end{aligned} \tag{5.61}$$

$$\angle\partial\Omega_{\alpha\beta} = \cap_{\alpha=1}^{4} \cap_{\beta=1}^{4} \partial\Omega_{\alpha\beta} = \{(z_1, z_2) | z_2 = 0, z_1 = 0\}. \tag{5.62}$$

The boundaries in the relative coordinates are independent of time. From such domains, boundaries, and vertex, the analytical conditions for the synchronization of the controlled slave systems and master systems can be developed using the theory of discontinuous dynamical systems. The domains, displacement and velocity boundaries, and vertex in relative phase space are also sketched in Fig. 5.4.

The controlled pendulum in domain Ω_α is in the relative coordinates

$$\dot{\mathbf{z}}^{(\alpha)} = \mathbf{g}^{(\alpha)}(\mathbf{z}^{(\alpha)}, \mathbf{x}, t) \text{ with } \dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, t), \tag{5.63}$$

where

$$\begin{aligned}
&\mathbf{g}^{(\alpha)}(\mathbf{z}^{(\alpha)}, \mathbf{x}, t) = (g_1^{(\alpha)}, g_2^{(\alpha)})^{\mathrm{T}},\\
&g_1^{(\alpha)}(\mathbf{z}^{(\alpha)}, \mathbf{x}, t) = z_2^{(\alpha)} - k_1 \text{ for } \alpha = 1, 2,\\
&g_1^{(\alpha)}(\mathbf{z}^{(\alpha)}, \mathbf{x}, t) = z_2^{(\alpha)} + k_1 \text{ for } \alpha = 3, 4,\\
&g_2^{(\alpha)}(\mathbf{z}^{(\alpha)}, \mathbf{x}, t) = \mathcal{G}(\mathbf{z}^{(\alpha)}, \mathbf{x}, t) - k_2 \text{ for } \alpha = 1, 4,\\
&g_2^{(\alpha)}(\mathbf{z}^{(\alpha)}, \mathbf{x}, t) = \mathcal{G}(\mathbf{z}^{(\alpha)}, \mathbf{x}, t) + k_2 \text{ for } \alpha = 2, 3
\end{aligned} \tag{5.64}$$

with

$$\begin{aligned} \mathcal{G}(\mathbf{z}^{(\alpha)}, \mathbf{x}, t) &= \dot{y}_2 - \dot{x}_2 \cos x_1 + x_2^2 \sin x_1 \\ &= -a_0 \sin(z_1^{(\alpha)} + \sin x_1) + Q_0 \cos \Omega t \\ &\quad + (d_1 x_2 - a_1 x_1 + a_2 x_1^3 - A_0 \cos \omega t) \cos x_1 + x_2^2 \sin x_1. \end{aligned} \tag{5.65}$$

The equation of motion on the boundary in the relative coordinates is

$$\dot{\mathbf{z}}^{(\alpha\beta)} = \mathbf{g}^{(\alpha\beta)}(\mathbf{z}^{(\alpha\beta)}, \mathbf{x}, t) \text{ with } \dot{\mathbf{x}} = \mathscr{F}(\mathbf{x}, t), \tag{5.66}$$

where

$$\begin{aligned} &\mathbf{g}^{(\alpha\beta)}(\mathbf{z}^{(\alpha\beta)}, \mathbf{x}, t) = (g_1^{(\alpha\beta)}, g_2^{(\alpha\beta)})^{\mathrm{T}}, \\ &g_1^{(\alpha\beta)}(\mathbf{z}^{(\alpha\beta)}, \mathbf{x}, t) = z_2 = 0 \text{ and } g_2^{(\alpha\beta)}(\mathbf{z}^{(\alpha\beta)}, \mathbf{x}, t) = 0 \end{aligned} \tag{5.67}$$

with

$$\begin{aligned} &z_1^{(\alpha\beta)} = 0 \text{ and } z_2^{(\alpha\beta)} = 0 \text{ on } \partial\Omega_{\alpha\beta} \text{ for } (\alpha, \beta) = (2,3), (1,4), \\ &z_1^{(\alpha\beta)} = C \text{ and } z_2^{(\alpha\beta)} = 0 \text{ on } \partial\Omega_{\alpha\beta} \text{ for } (\alpha, \beta) = (1,2), (3,4). \end{aligned} \tag{5.68}$$

5.3.1 Synchronization Dynamics

From the theory of the discontinuous dynamical system in Chap. 2 (e.g., [1, 2]), the synchronization dynamics of the controlled pendulum to the Duffing oscillator can be discussed. Thus, the G-functions in the relative coordinates for $\mathbf{z}_m \in \partial\Omega_{ij}$ at $t = t_m$ are given by

$$G_{\partial\Omega_{ij}}^{(\alpha)}(\mathbf{z}_m, \mathbf{x}, t_{m\pm}) = \mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot [\mathbf{g}^{(\alpha)}(\mathbf{z}_m, \mathbf{x}, t_{m\pm}) - \mathbf{g}^{(ij)}(\mathbf{z}_m, \mathbf{x}, t_{m\pm})], \tag{5.69}$$

$$G_{\partial\Omega_{ij}}^{(1,\alpha)}(\mathbf{z}_m, \mathbf{x}, t_{m\pm}) = \mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot [D\mathbf{g}^{(\alpha)}(\mathbf{z}_m, \mathbf{x}, t_{m\pm}) - D\mathbf{g}^{(ij)}(\mathbf{z}_m, \mathbf{x}, t_{m\pm})], \tag{5.70}$$

where $G_{\partial\Omega_{ij}}^{(\alpha)}(\mathbf{z}_m, \mathbf{x}, t_{m\pm})$ and $G_{\partial\Omega_{ij}}^{(1,\alpha)}(\mathbf{z}_m, \mathbf{x}, t_{m\pm})$ are the zero-order and first-order G-functions of the flow in the domain Ω_α ($\alpha \in \{i, j\}$) at the boundary $\partial\Omega_{ij}$ ($i, j = 1, 2, 3, 4$). The normal vectors of the relative boundaries from Eq. (5.61) are

$$\mathbf{n}_{\partial\Omega_{12}} = \mathbf{n}_{\partial\Omega_{34}} = (0, 1)^{\mathrm{T}} \text{ and } \mathbf{n}_{\partial\Omega_{23}} = \mathbf{n}_{\partial\Omega_{14}} = (1, 0)^{\mathrm{T}}. \tag{5.71}$$

The corresponding G-functions at the boundary $\partial\Omega_{ij}$ $(i, j = 1, 2, 3, 4)$ from Eqs. (6.63) to (6.67) are for domain Ω_α ($\alpha \in \{1, 2, 3, 4\}$),

$$\begin{aligned} &G^{(\alpha)}_{\partial\Omega_{12}}(\mathbf{z}_m,\mathbf{x},t_{m\pm}) = G^{(\alpha)}_{\partial\Omega_{34}}(\mathbf{z}_m,\mathbf{x},t_{m\pm}) = g^{(\alpha)}_2(\mathbf{z}_m,\mathbf{x},t_{m\pm}),\\ &G^{(\alpha)}_{\partial\Omega_{23}}(\mathbf{z}_m,\mathbf{x},t_{m\pm}) = G^{(\alpha)}_{\partial\Omega_{14}}(\mathbf{z}_m,\mathbf{x},t_{m\pm}) = g^{(\alpha)}_1(\mathbf{z}_m,\mathbf{x},t_{m\pm}),\end{aligned} \tag{5.72}$$

$$\begin{aligned} &G^{(1,\alpha)}_{\partial\Omega_{12}}(\mathbf{z}_m,\mathbf{x},t_{m\pm}) = G^{(1,\alpha)}_{\partial\Omega_{34}}(\mathbf{z}_m,\mathbf{x},t_{m\pm}) = Dg^{(\alpha)}_2(\mathbf{z}_m,\mathbf{x},t_{m\pm}),\\ &G^{(1,\alpha)}_{\partial\Omega_{23}}(\mathbf{z}_m,\mathbf{x},t_{m\pm}) = G^{(1,\alpha)}_{\partial\Omega_{14}}(\mathbf{z}_m,\mathbf{x},t_{m\pm}) = Dg^{(\alpha)}_1(\mathbf{z}_m,\mathbf{x},t_{m\pm}),\end{aligned} \tag{5.73}$$

where

$$\begin{aligned} Dg^{(\alpha)}_1(\mathbf{z}^{(\alpha)},\mathbf{x},t) &= g^{(\alpha)}_2(\mathbf{z}^{(\alpha)},\mathbf{x},t);\\ Dg^{(\alpha)}_2(\mathbf{z}^{(\alpha)},\mathbf{x},t) &= D\mathcal{G}(\mathbf{z}^{(\alpha)},\mathbf{x},t)\\ &= -a_0(z^{(\alpha)}_2 + x_2\cos x_1)\cos(z^{(\alpha)}_1 + \sin x_1) - Q_0\Omega\sin\Omega t\\ &\quad + (d_1F_2(\mathbf{x},t) - a_1x_2 + 3a_2x_1^2x_2 + \omega A_0\sin\omega t)\cos x_1\\ &\quad + (d_1x_2 - a_1x_1 + a_2x_1^3 - A_0\cos\omega t)x_2\sin x_1 + x_2^3\sin x_1. \end{aligned} \tag{5.74}$$

The G-functions in domains Ω_α $(\alpha\in\{1,2,3,4\})$ at the boundary $\partial\Omega_{ij}$ $(i,j=1,2,3,4)$ are

$$\begin{aligned} &G^{(\alpha)}_{\partial\Omega_{12}}(\mathbf{z}^{(\alpha)},\mathbf{x},t) = G^{(\alpha)}_{\partial\Omega_{34}}(\mathbf{z}^{(\alpha)},\mathbf{x},t) = g^{(\alpha)}_2(\mathbf{z}^{(\alpha)},\mathbf{x},t),\\ &G^{(\alpha)}_{\partial\Omega_{23}}(\mathbf{z}^{(\alpha)},\mathbf{x},t) = G^{(\alpha)}_{\partial\Omega_{14}}(\mathbf{z}^{(\alpha)},\mathbf{x},t) = g^{(\alpha)}_1(\mathbf{z}^{(\alpha)},\mathbf{x},t);\\ &G^{(1,\alpha)}_{\partial\Omega_{12}}(\mathbf{z}^{(\alpha)},\mathbf{x},t) = G^{(1,\alpha)}_{\partial\Omega_{34}}(\mathbf{z}^{(\alpha)},\mathbf{x},t) = Dg^{(\alpha)}_2(\mathbf{z}^{(\alpha)},\mathbf{x},t),\\ &G^{(1,\alpha)}_{\partial\Omega_{23}}(\mathbf{z}^{(\alpha)},\mathbf{x},t) = G^{(1,\alpha)}_{\partial\Omega_{14}}(\mathbf{z}^{(\alpha)},\mathbf{x},t) = Dg^{(\alpha)}_1(\mathbf{z}^{(\alpha)},\mathbf{x},t).\end{aligned} \tag{5.75}$$

(A) *Flow Switchability on the Separation Boundary*. From Chaps. 2–4, the analytical conditions of a flow sliding on the boundaries of $\partial\Omega_{12}$, $\partial\Omega_{34}$, $\partial\Omega_{23}$ and $\partial\Omega_{14}$ for the controlled pendulum are

$$\begin{aligned} &\left.\begin{aligned} &G^{(1)}_{\partial\Omega_{12}}(\mathbf{z}_m,\mathbf{x},t_{m-}) = g^{(1)}_2(\mathbf{z}_m,\mathbf{x},t_{m-}) < 0,\\ &G^{(2)}_{\partial\Omega_{12}}(\mathbf{z}_m,\mathbf{x},t_{m-}) = g^{(2)}_2(\mathbf{z}_m,\mathbf{x},t_{m-}) > 0 \end{aligned}\right\} \text{ for } \mathbf{z}_m\in\partial\Omega_{12};\\ &\left.\begin{aligned} &G^{(3)}_{\partial\Omega_{34}}(\mathbf{z}_m,\mathbf{x},t_{m-}) = g^{(3)}_2(\mathbf{z}_m,\mathbf{x},t_{m-}) > 0,\\ &G^{(4)}_{\partial\Omega_{34}}(\mathbf{z}_m,\mathbf{x},t_{m-}) = g^{(4)}_2(\mathbf{z}_m,\mathbf{x},t_{m-}) < 0 \end{aligned}\right\} \text{ for } \mathbf{z}_m\in\partial\Omega_{34}.\end{aligned} \tag{5.76}$$

$$\begin{aligned} &\left.\begin{aligned} &G^{(2)}_{\partial\Omega_{23}}(\mathbf{z}_m,\mathbf{x},t_{m-}) = g^{(2)}_1(\mathbf{z}_m,\mathbf{x},t_{m-}) < 0,\\ &G^{(3)}_{\partial\Omega_{12}}(\mathbf{z}_m,\mathbf{x},t_{m-}) = g^{(3)}_1(\mathbf{z}_m,\mathbf{x},t_{m-}) > 0 \end{aligned}\right\} \text{ for } \mathbf{z}_m\in\partial\Omega_{23};\\ &\left.\begin{aligned} &G^{(1)}_{\partial\Omega_{14}}(\mathbf{z}_m,\mathbf{x},t_{m-}) = g^{(1)}_1(\mathbf{z}_m,\mathbf{x},t_{m-}) < 0,\\ &G^{(4)}_{\partial\Omega_{14}}(\mathbf{z}_m,\mathbf{x},t_{m-}) = g^{(4)}_1(\mathbf{z}_m,\mathbf{x},t_{m-}) > 0 \end{aligned}\right\} \text{ for } \mathbf{z}_m\in\partial\Omega_{14}.\end{aligned} \tag{5.77}$$

The analytical conditions of a flow passing through the boundaries $\partial\Omega_{ij}$ $(i,j=1,2,3,4;\ j\neq i)$ for the controlled pendulum are

$$\left.\begin{aligned}&G^{(1)}_{\partial\Omega_{12}}(\mathbf{z}_m,\mathbf{x},t_{m-})=g^{(1)}_2(\mathbf{z}_m,\mathbf{x},t_{m-})<0,\\&G^{(2)}_{\partial\Omega_{12}}(\mathbf{z}_m,\mathbf{x},t_{m+})=g^{(2)}_2(\mathbf{z}_m,\mathbf{x},t_{m+})<0\end{aligned}\right\}\text{ for }\mathbf{z}_m\in\partial\Omega_{12};$$
$$\left.\begin{aligned}&G^{(3)}_{\partial\Omega_{34}}(\mathbf{z}_m,\mathbf{x},t_{m-})=g^{(3)}_2(\mathbf{z}_m,\mathbf{x},t_{m-})>0,\\&G^{(4)}_{\partial\Omega_{34}}(\mathbf{z}_m,\mathbf{x},t_{m+})=g^{(4)}_2(\mathbf{z}_m,\mathbf{x},t_{m+})>0\end{aligned}\right\}\text{ for }\mathbf{z}_m\in\partial\Omega_{34}. \tag{5.78}$$

$$\left.\begin{aligned}&G^{(2)}_{\partial\Omega_{23}}(\mathbf{z}_m,\mathbf{x},t_{m-})=g^{(2)}_1(\mathbf{z}_m,\mathbf{x},t_{m-})<0,\\&G^{(3)}_{\partial\Omega_{23}}(\mathbf{z}_m,\mathbf{x},t_{m+})=g^{(3)}_1(\mathbf{z}_m,\mathbf{x},t_{m+})<0\end{aligned}\right\}\text{ for }\mathbf{z}_m\in\partial\Omega_{23};$$
$$\left.\begin{aligned}&G^{(1)}_{\partial\Omega_{14}}(\mathbf{z}_m,\mathbf{x},t_{m-})=g^{(1)}_1(\mathbf{z}_m,\mathbf{x},t_{m-})>0,\\&G^{(4)}_{\partial\Omega_{14}}(\mathbf{z}_m,\mathbf{x},t_{m+})=g^{(4)}_1(\mathbf{z}_m,\mathbf{x},t_{m+})>0\end{aligned}\right\}\text{ for }\mathbf{z}_m\in\partial\Omega_{14}. \tag{5.79}$$

The analytical conditions of a flow grazing to the boundaries $\partial\Omega_{ij}$ $(i,j=1,2,3,4;\ j\neq i)$ for the controlled pendulum are

$$\begin{aligned}&\left.\begin{aligned}&G^{(\alpha)}_{\partial\Omega_{12}}(\mathbf{z}_m,\mathbf{x},t_{m\pm})=g^{(\alpha)}_2(\mathbf{z}_m,\mathbf{x},t_{m\pm})=0,\\&(-1)^{\alpha}G^{(1,\alpha)}_{\partial\Omega_{12}}(\mathbf{z}_m,\mathbf{x},t_{m\pm})=(-1)^{\alpha}Dg^{(\alpha)}_2(\mathbf{z}_m,\mathbf{x},t_{m\pm})<0\end{aligned}\right\}\\&\quad\text{for }\mathbf{z}_m\in\partial\Omega_{12}\text{ in }\Omega_\alpha(\alpha\in\{1,2\});\\&\left.\begin{aligned}&G^{(\alpha)}_{\partial\Omega_{34}}(\mathbf{z}_m,\mathbf{x},t_{m\pm})=g^{(\alpha)}_2(\mathbf{z}_m,\mathbf{x},t_{m\pm})=0,\\&(-1)^{\alpha}G^{(1,\alpha)}_{\partial\Omega_{34}}(\mathbf{z}_m,\mathbf{x},t_{m\pm})=(-1)^{\alpha}Dg^{(\alpha)}_2(\mathbf{z}_m,\mathbf{x},t_{m\pm})>0\end{aligned}\right\}\\&\quad\text{for }\mathbf{z}_m\in\partial\Omega_{34}\text{ in }\Omega_\alpha(\alpha\in\{3,4\});\end{aligned} \tag{5.80}$$

$$\begin{aligned}&\left.\begin{aligned}&G^{(\alpha)}_{\partial\Omega_{23}}(\mathbf{z}_m,\mathbf{x},t_{m\pm})=g^{(\alpha)}_1(\mathbf{z}_m,\mathbf{x},t_{m\pm})=0\\&(-1)^{\alpha}G^{(1,\alpha)}_{\partial\Omega_{23}}(\mathbf{z}_m,\mathbf{x},t_{m\pm})=(-1)^{\alpha}Dg^{(\alpha)}_1(\mathbf{z}_m,\mathbf{x},t_{m\pm})>0\end{aligned}\right\}\\&\quad\text{for }\mathbf{z}_m\in\partial\Omega_{23}\text{ in }\Omega_\alpha(\alpha\in\{2,3\});\\&\left.\begin{aligned}&G^{(\alpha)}_{\partial\Omega_{14}}(\mathbf{z}_m,\mathbf{x},t_{m\pm})=g^{(\alpha)}_1(\mathbf{z}_m,\mathbf{x},t_{m\pm})=0\\&(-1)^{\alpha}G^{(1,\alpha)}_{\partial\Omega_{14}}(\mathbf{z}_m,\mathbf{x},t_{m\pm})=(-1)^{\alpha}Dg^{(\alpha)}_1(\mathbf{z}_m,\mathbf{x},t_{m\pm})<0\end{aligned}\right\}\\&\quad\text{for }\mathbf{z}_m\in\partial\Omega_{14}\text{ in }\Omega_\alpha(\alpha\in\{1,4\}).\end{aligned} \tag{5.81}$$

The analytical conditions for onset of a sliding flow on the boundaries $\partial\Omega_{ij}$ (i, j = 1, 2, 3, 4; j ≠ i) for the controlled pendulum are

$$
\left.\begin{aligned}
&G^{(1)}_{\partial\Omega_{12}}(\mathbf{z}_m,\mathbf{x},t_{m-}) = g^{(1)}_2(\mathbf{z}_m,\mathbf{x},t_{m-}) < 0,\\
&G^{(2)}_{\partial\Omega_{12}}(\mathbf{z}_m,\mathbf{x},t_{m\pm}) = g^{(2)}_2(\mathbf{z}_m,\mathbf{x},t_{m\pm}) = 0,\\
&G^{(1,2)}_{\partial\Omega_{12}}(\mathbf{z}_m,\mathbf{x},t_{m\pm}) = Dg^{(2)}_2(\mathbf{z}_m,\mathbf{x},t_{m\pm}) < 0
\end{aligned}\right\}\text{ from } \Omega_1 \to \partial\Omega_{12};
$$

$$
\left.\begin{aligned}
&G^{(3)}_{\partial\Omega_{34}}(\mathbf{z}_m,\mathbf{x},t_{m-}) = g^{(3)}_2(\mathbf{z}_m,\mathbf{x},t_{m-}) > 0,\\
&G^{(4)}_{\partial\Omega_{34}}(\mathbf{z}_m,\mathbf{x},t_{m\pm}) = g^{(4)}_2(\mathbf{z}_m,\mathbf{x},t_{m\pm}) = 0,\\
&G^{(1,4)}_{\partial\Omega_{34}}(\mathbf{z}_m,\mathbf{x},t_{m\pm}) = Dg^{(4)}_2(\mathbf{z}_m,\mathbf{x},t_{m\pm}) > 0
\end{aligned}\right\}\text{ from } \Omega_3 \to \partial\Omega_{34}. \tag{5.82}
$$

$$
\left.\begin{aligned}
&G^{(2)}_{\partial\Omega_{23}}(\mathbf{z}_m,\mathbf{x},t_{m-}) = g^{(2)}_1(\mathbf{z}_m,\mathbf{x},t_{m-}) < 0,\\
&G^{(3)}_{\partial\Omega_{23}}(\mathbf{z}_m,\mathbf{x},t_{m\pm}) = g^{(3)}_1(\mathbf{z}_m,\mathbf{x},t_{m\pm}) = 0,\\
&G^{(1,3)}_{\partial\Omega_{23}}(\mathbf{z}_m,\mathbf{x},t_{m\pm}) = Dg^{(3)}_1(\mathbf{z}_m,\mathbf{x},t_{m\pm}) < 0
\end{aligned}\right\}\text{ from } \Omega_2 \to \partial\Omega_{23};
$$

$$
\left.\begin{aligned}
&G^{(4)}_{\partial\Omega_{14}}(\mathbf{z}_m,\mathbf{x},t_{m-}) = g^{(4)}_1(\mathbf{z}_m,\mathbf{x},t_{m-}) > 0,\\
&G^{(1)}_{\partial\Omega_{14}}(\mathbf{z}_m,\mathbf{x},t_{m\pm}) = g^{(1)}_1(\mathbf{z}_m,\mathbf{x},t_{m\pm}) = 0,\\
&G^{(1,1)}_{\partial\Omega_{14}}(\mathbf{z}_m,\mathbf{x},t_{m\pm}) = Dg^{(1)}_1(\mathbf{z}_m,\mathbf{x},t_{m\pm}) > 0
\end{aligned}\right\}\text{ from } \Omega_4 \to \partial\Omega_{14}. \tag{5.83}
$$

The analytical conditions for vanishing of a sliding flow on the boundaries $\partial\Omega_{ij}$ $(i,j = 1,2,3,4;\ j \neq i)$ to a domain for the controlled pendulum are

$$
\left.\begin{aligned}
&(-1)^{\beta} G^{(\beta)}_{\partial\Omega_{12}}(\mathbf{z}_m,\mathbf{x},t_{m-}) = (-1)^{\beta} g^{(\beta)}_2(\mathbf{z}_m,\mathbf{x},t_{m-}) > 0\\
&G^{(\alpha)}_{\partial\Omega_{12}}(\mathbf{z}_m,\mathbf{x},t_{m\mp}) = g^{(\alpha)}_2(\mathbf{z}_m,\mathbf{x},t_{m\mp}) = 0\\
&(-1)^{\alpha} G^{(1,\alpha)}_{\partial\Omega_{12}}(\mathbf{z}_m,\mathbf{x},t_{m\mp}) = (-1)^{\alpha} Dg^{(\alpha)}_2(\mathbf{z}_m,\mathbf{x},t_{m\mp}) < 0,\\
&\text{for } \mathbf{z}_m \in \partial\Omega_{12};\, \alpha,\beta \in \{1,2\} \text{ and } \beta \neq \alpha
\end{aligned}\right\}\text{ from } \partial\Omega_{12} \to \Omega_{\alpha}; \tag{5.84}
$$

$$
\left.\begin{aligned}
&(-1)^{\beta} G^{(\beta)}_{\partial\Omega_{34}}(\mathbf{z}_m,\mathbf{x},t_{m-}) = (-1)^{\beta} g^{(\beta)}_2(\mathbf{z}_m,\mathbf{x},t_{m-}) < 0,\\
&G^{(\alpha)}_{\partial\Omega_{34}}(\mathbf{z}_m,\mathbf{x},t_{m\mp}) = g^{(\alpha)}_2(\mathbf{z}_m,\mathbf{x},t_{m\mp}) = 0,\\
&(-1)^{\alpha} G^{(1,\alpha)}_{\partial\Omega_{34}}(\mathbf{z}_m,\mathbf{x},t_{m\mp}) = (-1)^{\alpha} Dg^{(\alpha)}_2(\mathbf{z}_m,\mathbf{x},t_{m\mp}) > 0\\
&\text{for } \mathbf{z}_m \in \partial\Omega_{34};\, \alpha,\beta \in \{3,4\} \text{ and } \beta \neq \alpha
\end{aligned}\right\}\text{ from } \partial\Omega_{34} \to \Omega_{\alpha}; \tag{5.85}
$$

$$
\left.\begin{aligned}
&(-1)^{\beta} G^{(\beta)}_{\partial\Omega_{23}}(\mathbf{z}_m,\mathbf{x},t_{m-}) = (-1)^{\beta} g^{(\beta)}_1(\mathbf{z}_m,\mathbf{x},t_{m-}) < 0,\\
&G^{(\alpha)}_{\partial\Omega_{23}}(\mathbf{z}_m,\mathbf{x},t_{m\mp}) = g^{(\alpha)}_1(\mathbf{z}_m,\mathbf{x},t_{m\mp}) = 0,\\
&(-1)^{\alpha} G^{(1,\alpha)}_{\partial\Omega_{23}}(\mathbf{z}_m,\mathbf{x},t_{m\mp}) = (-1)^{\alpha} Dg^{(\alpha)}_1(\mathbf{z}_m,\mathbf{x},t_{m\mp}) > 0\\
&\text{for } \mathbf{z}_m \in \partial\Omega_{23};\, \alpha,\beta \in \{2,3\} \text{ and } \beta \neq \alpha
\end{aligned}\right\}\text{ from } \partial\Omega_{23} \to \Omega_{\alpha}; \tag{5.86}
$$

$$\left.\begin{aligned}&(-1)^{\beta}G^{(\beta)}_{\partial\Omega_{23}}(\mathbf{z}_m,\mathbf{x},t_{m-})=(-1)^{\beta}g^{(\beta)}_1(\mathbf{z}_m,\mathbf{x},t_{m-})>0,\\&G^{(\alpha)}_{\partial\Omega_{23}}(\mathbf{z}_m,\mathbf{x},t_{m\mp})=g^{(\alpha)}_1(\mathbf{z}_m,\mathbf{x},t_{m\mp})=0,\\&(-1)^{\alpha}G^{(1,\alpha)}_{\partial\Omega_{23}}(\mathbf{z}_m,\mathbf{x},t_{m\mp})=(-1)^{\alpha}Dg^{(\alpha)}_1(\mathbf{z}_m,\mathbf{x},t_{m\mp})<0\\&\text{for }\mathbf{z}_m\in\partial\Omega_{14};\alpha,\beta\in\{1,4\}\text{ and }\beta\neq\alpha\end{aligned}\right\}\text{from }\partial\Omega_{14}\to\Omega_{\alpha}.\quad(5.87)$$

(B) *Synchronization Conditions*. The analytical conditions for the complete synchronization of the controlled pendulum with the Duffing oscillator at the intersection of the two boundaries ($\mathbf{z}_m=\mathbf{0}$) are given by

$$\begin{aligned}&\left.\begin{aligned}&G^{(1)}_{\partial\Omega_{14}}(\mathbf{z}_m,\mathbf{x},t_{m-})=g^{(1)}_1(\mathbf{z}_m,\mathbf{x},t_{m-})<0,\\&G^{(1)}_{\partial\Omega_{12}}(\mathbf{z}_m,\mathbf{x},t_{m-})=g^{(1)}_2(\mathbf{z}_m,\mathbf{x},t_{m-})<0\end{aligned}\right\}\text{for }\mathbf{z}_m\in\partial\Omega_{12}\cap\partial\Omega_{14}\text{ on }\Omega_1;\\&\left.\begin{aligned}&G^{(2)}_{\partial\Omega_{12}}(\mathbf{z}_m,\mathbf{x},t_{m-})=g^{(2)}_2(\mathbf{z}_m,\mathbf{x},t_{m-})>0,\\&G^{(2)}_{\partial\Omega_{23}}(\mathbf{z}_m,\mathbf{x},t_{m-})=g^{(2)}_1(\mathbf{z}_m,\mathbf{x},t_{m-})<0\end{aligned}\right\}\text{for }\mathbf{z}_m\in\partial\Omega_{12}\cap\partial\Omega_{23}\text{ on }\Omega_2;\\&\left.\begin{aligned}&G^{(3)}_{\partial\Omega_{23}}(\mathbf{z}_m,\mathbf{x},t_{m-})=g^{(3)}_1(\mathbf{z}_m,\mathbf{x},t_{m-})>0,\\&G^{(3)}_{\partial\Omega_{34}}(\mathbf{z}_m,\mathbf{x},t_{m-})=g^{(3)}_2(\mathbf{z}_m,\mathbf{x},t_{m-})>0\end{aligned}\right\}\text{for }\mathbf{z}_m\in\partial\Omega_{23}\cap\partial\Omega_{34}\text{ on }\Omega_3;\\&\left.\begin{aligned}&G^{(4)}_{\partial\Omega_{34}}(\mathbf{z}_m,\mathbf{x},t_{m-})=g^{(4)}_2(\mathbf{z}_m,\mathbf{x},t_{m-})<0,\\&G^{(4)}_{\partial\Omega_{14}}(\mathbf{z}_m,\mathbf{x},t_{m-})=g^{(4)}_1(\mathbf{z}_m,\mathbf{x},t_{m-})>0\end{aligned}\right\}\text{for }\mathbf{z}_m\in\partial\Omega_{34}\cap\partial\Omega_{14}\text{ on }\Omega_4.\end{aligned}\quad(5.88)$$

Four basic functions from Eq. (5.63) are defined as

$$\begin{aligned}&g_1(\mathbf{z}^{(\alpha)},\mathbf{x},t)\equiv g^{(\alpha)}_1(\mathbf{z}^{(\alpha)},\mathbf{x},t)=z^{(\alpha)}_2-k_1\text{ in }\Omega_\alpha\text{ for }\alpha=1,2;\\&g_2(\mathbf{z}^{(\alpha)},\mathbf{x},t)\equiv g^{(\alpha)}_1(\mathbf{z}^{(\alpha)},\mathbf{x},t)=z^{(\alpha)}_2+k_1\text{ in }\Omega_\alpha\text{ for }\alpha=3,4;\\&g_3(\mathbf{z}^{(\alpha)},\mathbf{x},t)\equiv g^{(\alpha)}_2(\mathbf{z}^{(\alpha)},\mathbf{x},t)=\mathcal{G}(\mathbf{z}^{(\alpha)},\mathbf{x},t)-k_2\text{ in }\Omega_\alpha\text{ for }\alpha=1,4;\\&g_4(\mathbf{z}^{(\alpha)},\mathbf{x},t)\equiv g^{(\alpha)}_2(\mathbf{z}^{(\alpha)},\mathbf{x},t)=\mathcal{G}(\mathbf{z}^{(\alpha)},\mathbf{x},t)+k_2\text{ in }\Omega_\alpha\text{ for }\alpha=2,3.\end{aligned}\quad(5.89)$$

The synchronization conditions in Eq. (5.88) become

$$\begin{aligned}&g_1(\mathbf{z}_m,\mathbf{x},t_{m-})=z_{2m}-k_1<0,\\&g_2(\mathbf{z}_m,\mathbf{x},t_{m-})=z_{2m}+k_1>0,\\&g_3(\mathbf{z}_m,\mathbf{x},t_{m-})=\mathcal{G}(\mathbf{z}_m,\mathbf{x},t_{m-})-k_2<0,\\&g_4(\mathbf{z}_m,\mathbf{x},t_{m-})=\mathcal{G}(\mathbf{z}_m,\mathbf{x},t_{m-})+k_2>0.\end{aligned}\quad(5.90)$$

For $\mathbf{z}_m=\mathbf{0}$, the synchronization conditions of the controlled pendulum with the Duffing oscillator with sinusoidal constraint in Eq. (5.38) are

$$\begin{aligned}
g_1(\mathbf{z}_m, \mathbf{x}, t_{m-}) &= -k_1 < 0,\\
g_2(\mathbf{z}_m, \mathbf{x}, t_{m-}) &= +k_1 > 0,\\
g_3(\mathbf{z}_m, \mathbf{x}, t_{m-}) &= \mathcal{G}(\mathbf{x}, t_{m-}) - k_2 < 0,\\
g_4(\mathbf{z}_m, \mathbf{x}, t_{m-}) &= \mathcal{G}(\mathbf{x}, t_{m-}) + k_2 > 0,
\end{aligned} \tag{5.91}$$

where

$$\begin{aligned}
\mathcal{G}(\mathbf{x}, t) = &-a_0 \sin(\sin x_1) + Q_0 \cos \Omega t\\
&+ (d_1 x_2 - a_1 x_1 + a_2 x_1^3 - A_0 \cos \omega t) \cos x_1 + x_2^2 \sin x_1.
\end{aligned} \tag{5.92}$$

The first two equations of Eq. (5.90) are satisfied if $k_1 > 0$ and $k_2 > 0$. The synchronization invariant set is given by the third and fourth equations, i.e.,

$$-k_2 < \mathcal{G}(\mathbf{x}, t_{m-}) < k_2. \tag{5.93}$$

In the small neighborhood of the synchronization of $\mathbf{z}_m = \mathbf{0}$, the attractivity conditions can be obtained for $||\mathbf{z} - \mathbf{z}_m|| < \varepsilon$, i.e.,

$$\begin{aligned}
0 \le z_2 < k_1 &\text{ and } \mathcal{G}(\mathbf{z}, \mathbf{x}, t) < k_2 \quad \text{for } z_1 \in [0, \infty) \text{ in } \Omega_1,\\
0 \le z_2 < k_1 &\text{ and } -k_2 < \mathcal{G}(\mathbf{z}, \mathbf{x}, t) \quad \text{for } z_1 \in [0, \infty) \text{ in } \Omega_2,\\
-k_1 < z_2 \le 0 &\text{ and } -k_2 < \mathcal{G}(\mathbf{z}, \mathbf{x}, t) \quad \text{for } z_1 \in (-\infty, 0] \text{ in } \Omega_3,\\
-k_1 < z_2 \le 0 &\text{ and } \mathcal{G}(\mathbf{z}, \mathbf{x}, t) < k_2 \quad \text{for } z_1 \in (-\infty, 0] \text{ in } \Omega_4
\end{aligned} \tag{5.94}$$

from which z_1^* and z_2^* are obtained. The initial conditions for the controlled pendulum synchronizing with the Duffing oscillator are determined by

$$y_1 = z_1^* + \sin x_1 \text{ and } y_2 = z_2^* + x_2 \cos x_1. \tag{5.95}$$

From a sliding flow vanishing on the boundary, the synchronization vanishing conditions at $\mathbf{z}^{(\alpha)}(t_{m\mp}) = \mathbf{z}_m^{(\alpha)} = \mathbf{z}_m$ are

$$\left.\begin{aligned}
g_1(\mathbf{z}_m^{(\alpha)}, \mathbf{x}, t_{m\mp}) &= z_{2m}^{(\alpha)} - k_1 = 0,\\
Dg_1(\mathbf{z}_m^{(\alpha)}, \mathbf{x}, t_{m\mp}) &= \mathcal{G}(\mathbf{z}_m^{(\alpha)}, \mathbf{x}, t_{m\mp}) > 0,\\
g_2(\mathbf{z}_m^{(\beta)}, \mathbf{x}, t_{m-}) &= z_{2m}^{(\beta)} + k_1 > 0
\end{aligned}\right\} \text{ for } (\alpha, \beta) \in \{(1,4), (2,3)\} \tag{5.96}$$

from $z_{m+\varepsilon} = y_1 - x_1 > 0$, and

$$\left.\begin{aligned}
g_1(\mathbf{z}_m^{(\alpha)}, \mathbf{x}, t_{m-}) &= z_{2m}^{(\alpha)} - k_1 < 0;\\
g_2(\mathbf{z}_m^{(\beta)}, \mathbf{x}, t_{m\mp}) &= z_{2m}^{(\beta)} + k_1 = 0,\\
Dg_2(\mathbf{z}_m^{(\beta)}, \mathbf{x}, t_{m\mp}) &= \mathcal{G}(\mathbf{z}_m^{(\beta)}, \mathbf{x}, t_{m\mp}) < 0
\end{aligned}\right\} \text{ for } (\alpha, \beta) \in \{(1,4), (2,3)\} \tag{5.97}$$

from $z_{m+\varepsilon} = y_1 - x_1 < 0$.

The vanishing conditions of synchronization are for $\mathbf{z}^{(\alpha)}(t_{m\mp}) = \mathbf{z}_m^{(\alpha)} = \mathbf{z}_m$

$$\left.\begin{aligned} g_3(\mathbf{z}_m^{(\alpha)}, \mathbf{x}, t_{m\mp}) &= \mathcal{G}(\mathbf{z}_m^{(\alpha)}, \mathbf{x}, t_{m\mp}) - k_2 = 0, \\ Dg_3(\mathbf{z}_m^{(\alpha)}, \mathbf{x}, t_{m\mp}) &= D\mathcal{G}(\mathbf{z}_m^{(\alpha)}, \mathbf{x}, t_{m\mp}) > 0; \\ g_4(\mathbf{z}_m^{(\beta)}, \mathbf{x}, t_{m-}) &= \mathcal{G}(\mathbf{z}_m^{(\beta)}, \mathbf{x}, t_{m-}) + k_2 > 0 \end{aligned}\right\} \text{for } (\alpha, \beta) \in \{(1,2),(4,3)\} \tag{5.98}$$

from $\dot{z}_{m+\varepsilon} = y_2 - x_2 > 0$, and

$$\left.\begin{aligned} g_3(\mathbf{z}_m^{(\alpha)}, \mathbf{x}, t_{m-}) &= \mathcal{G}(\mathbf{z}_m^{(\alpha)}, \mathbf{x}, t_{m-}) - k_2 < 0; \\ g_4(\mathbf{z}_m^{(\beta)}, \mathbf{x}, t_{m\mp}) &= \mathcal{G}(\mathbf{z}_m^{(\beta)}, \mathbf{x}, t_{m\mp}) + k_2 = 0, \\ g_4(\mathbf{z}_m^{(\beta)}, \mathbf{x}, t_{m\mp}) &= D\mathcal{G}(\mathbf{z}_m^{(\beta)}, \mathbf{x}, t_{m\mp}) < 0 \end{aligned}\right\} \text{for } (\alpha, \beta) \in \{(1,2),(4,3)\} \tag{5.99}$$

from $\dot{z}_{m+\varepsilon} = y_2 - x_2 < 0$.

From the sliding flow appearance on the boundary, the synchronization onset conditions at $\mathbf{z}^{(\alpha)}(t_{m\mp}) = \mathbf{z}_m^{(\alpha)} = \mathbf{z}_m$ are

$$\left.\begin{aligned} g_1(\mathbf{z}_m^{(\alpha)}, \mathbf{x}, t_{m\pm}) &= z_{2m}^{(\alpha)} - k_1 = 0, \\ Dg_1(\mathbf{z}_m^{(\alpha)}, \mathbf{x}, t_{m\pm}) &= \mathcal{G}(\mathbf{z}_m^{(\alpha)}, \mathbf{x}, t_{m\pm}) > 0, \\ g_2(\mathbf{z}_m^{(\beta)}, \mathbf{x}, t_{m-}) &= z_{2m}^{(\beta)} + k_1 > 0; \end{aligned}\right\} \text{for } (\alpha, \beta) \in \{(1,4),(2,3)\} \tag{5.100}$$

from $z_{m-\varepsilon} = y_1 - x_1 > 0$, and

$$\left.\begin{aligned} g_1(\mathbf{z}_m^{(\alpha)}, \mathbf{x}, t_{m-}) &= z_{2m}^{(\alpha)} - k_1 < 0; \\ g_2(\mathbf{z}_m^{(\beta)}, \mathbf{x}, t_{m\pm}) &= z_{2m}^{(\beta)} + k_1 = 0, \\ Dg_2(\mathbf{z}_m^{(\beta)}, \mathbf{x}, t_{m\pm}) &= \mathcal{G}(\mathbf{z}_m^{(\beta)}, \mathbf{x}, t_{m\pm}) < 0 \end{aligned}\right\} \text{for } (\alpha, \beta) \in \{(1,4),(2,3)\} \tag{5.101}$$

from $z_{m+\varepsilon} = y_1 - x_1 < 0$.

The synchronization onset conditions for $\mathbf{z}^{(\alpha)}(t_{m\pm}) = \mathbf{z}_m^{(\alpha)} = \mathbf{z}_m$ are

$$\left.\begin{aligned} g_3(\mathbf{z}_m^{(\alpha)}, \mathbf{x}, t_{m\pm}) &= \mathcal{G}(\mathbf{z}_m^{(\alpha)}, \mathbf{x}, t_{m\pm}) - k_2 = 0, \\ Dg_3(\mathbf{z}_m^{(\alpha)}, \mathbf{x}, t_{m\pm}) &= D\mathcal{G}(\mathbf{z}_m^{(\alpha)}, \mathbf{x}, t_{m\pm}) > 0; \\ g_4(\mathbf{z}_m^{(\beta)}, \mathbf{x}, t_{m-}) &= \mathcal{G}(\mathbf{z}_m^{(\beta)}, \mathbf{x}, t_{m-}) + k_2 > 0 \end{aligned}\right\} \text{for } (\alpha, \beta) \in \{(1,2),(4,3)\} \tag{5.102}$$

from $\dot{z}_{m-\varepsilon} = y_2 - x_2 > 0$, and

$$\left.\begin{aligned} g_3(\mathbf{z}_m^{(\alpha)}, \mathbf{x}, t_{m-}) &= \mathcal{G}(\mathbf{z}_m^{(\alpha)}, \mathbf{x}, t_{m-}) - k_2 < 0; \\ g_4(\mathbf{z}_m^{(\beta)}, \mathbf{x}, t_{m\pm}) &= \mathcal{G}(\mathbf{z}_m^{(\beta)}, \mathbf{x}, t_{m\pm}) + k_2 = 0, \\ g_4(\mathbf{z}_m^{(\beta)}, \mathbf{x}, t_{m\pm}) &= D\mathcal{G}(\mathbf{z}_m^{(\beta)}, \mathbf{x}, t_{m\pm}) < 0 \end{aligned}\right\} \text{for } (\alpha, \beta) \in \{(1,2),(4,3)\} \tag{5.103}$$

from $\dot{z}_{m-\varepsilon} = y_2 - x_2 < 0$.

5.3.2 Sinusoidal Synchronization of Chaotic Motions

As in Min and Luo [5, 6], numerical simulations are presented for a better understanding of the function synchronization of pendulum and Duffing oscillator. For illustration, consider the following system parameters:

$$\begin{aligned} &a_1 = a_2 = 1.0, d_1 = 0.25, A_0 = 0.4, \omega = 1.0 \\ &a_0 = 1.0, Q_0 = 0.275, \Omega = 2.18519 \end{aligned} \tag{5.104}$$

Consider the initial conditions $(x_1, x_2) = (-1.31892, 0.10879)$ and $(y_1, y_2) = (-0.96845,\ 0.02711)$, the Duffing oscillator exhibits chaotic motion, and the pendulum system has the chaotic motion. The constraint conditions ($y_1 = \sin x_1$ and $y_2 = x_2 \cos x_1$) are adopted. The sinusoidal chaotic synchronization scenario for the controlled pendulum and the Duffing oscillator is presented in Fig. 5.5 with $k_1 = 1$ via switching points versus control parameters. $y_{1k} = \sin x_{1k}$ and $y_{2k} = x_{2k}$ $\cos x_{1k}$. The acronyms "FS", "PS", and "NS" represent "*f*ull synchronization", "*p*artial synchronization", and "*n*on-synchronization", respectively. "A" and "V" denote synchronization appearance and vanishing. The switching displacement, switching velocity, and switching phases for synchronization of the controlled pendulum with the Duffing oscillator are illustrated in Fig. 5.5a–e, respectively. From such synchronization scenario, the partial, sinusoidal, chaotic synchronization of the controlled pendulum with the Duffing oscillator is in the range of $k_2 \in (0.017, 1.330)$. If $k_2 \in (0, 0.017)$, no sinusoidal, chaotic synchronization between the two systems can be obtained. If $k_2 \in (1.330, \infty)$, the full, sinusoidal, chaotic synchronization of such two systems is achieved. For a global view of synchronization, a control parameter map (k_1, k_2) is presented in Fig. 5.6. The shaded area is a partial sinusoidal synchronization zone. For $k_2 > 1.330$ and $k_1 \neq 0$, the sinusoidal synchronization of the two systems exists. For small k_2, the non-synchronization area is observed. The bottom boundary of the partial synchronization in the control parameter map is zigzagged because chaotic motions in the Duffing oscillator are synchronized by the controlled chaotic pendulum.

From the parameter map, for $(k_1, k_2) = (1.0, 0.8)$, the partial sinusoidal synchronization of the controlled pendulum and the Duffing oscillator can be observed. The time-histories of displacements, velocities, and G-functions plus trajectories in phase planes are illustrated in Fig. 5.7a–d, respectively. The responses of the controlled pendulum as the controlled slave system are given by dashed curves, and the responses of the Duffing oscillator as a master system are presented by solid curves. The hollow and solid circular symbols are for synchronization appearance and vanishing, respectively. The sinusoidal synchronization and non-synchronization of the displacements and velocities for the two oscillators are presented in Fig. 5.7a, b. The shaded portions are for synchronization, and the rest portions are for non-synchronization. Compared to identical synchronization, the aforementioned two plots, it is very difficult to see the sinusoidal chaotic synchronization like the identical synchronization because the synchronization is based on a sinusoidal function. This is because the sinusoidal synchronization is inserted between the Duffing oscillator and the pendulum. Thus, the time-histories of G-functions should be presented to determine the synchronicity, as shown in Fig. 5.7c. The non-shaded regions are for non-synchronization, and the G-functions for non-synchronization are presented by dashed curves. The G-function tells synchronicity between the two oscillators under the sinusoidal

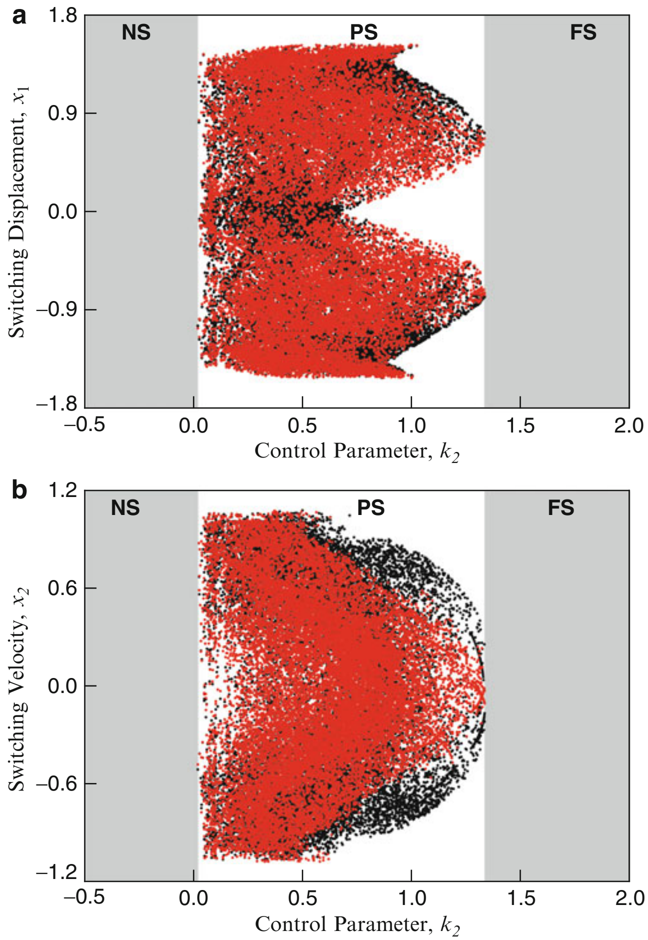

Fig. 5.5 Sinusoidal chaotic synchronization scenario of switching points versus control parameter k_2: (**a**) switching displacement and (**b**) switching velocity of master system; (**c**) switching displacement and (**d**) switching velocity of slave system; (**e**) switching phase. (Control parameter: $k_1 = 1$. Duffing: $a_1 = a_2 = 1.0$, $d_1 = 0.25$, $A_0 = 0.4$, $\omega = 1.0$. Pendulum: $a_0 = 1.0$, $Q_0 = 0.275$, $\Omega = 2.18519$) (*FS* Full synchronization, *PS* Partial synchronization, *NS* Non-synchronization)

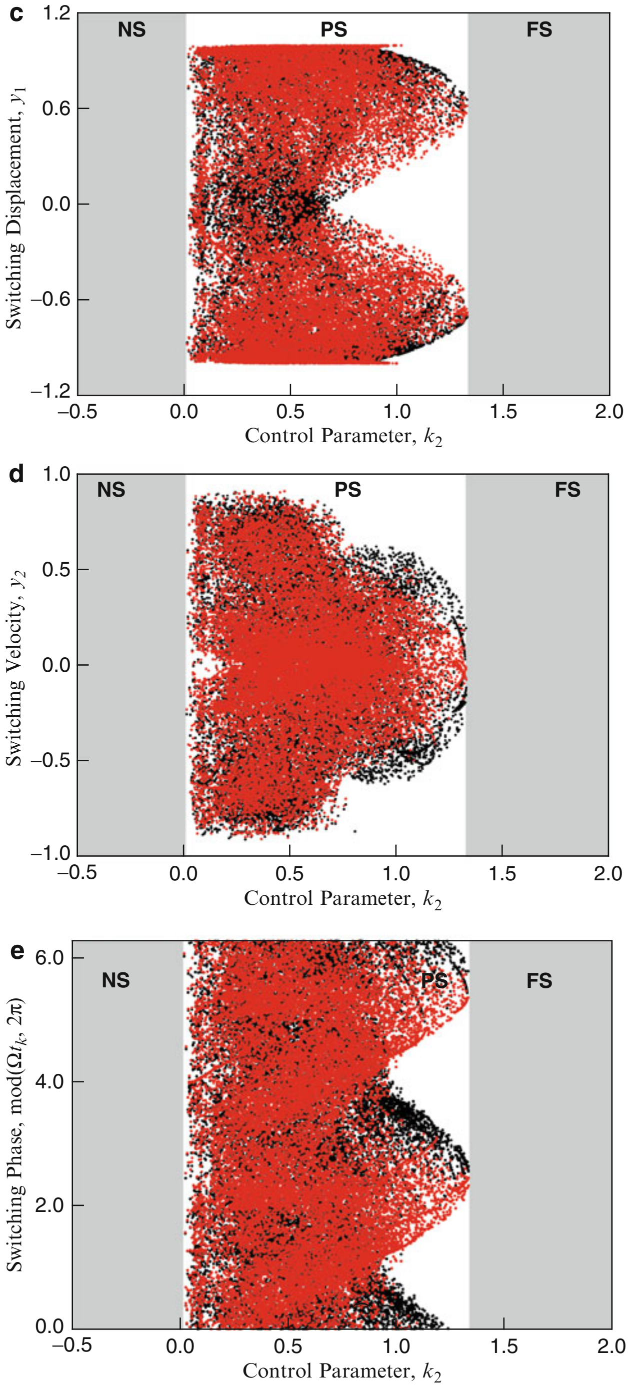

Fig. 5.5 (continued)

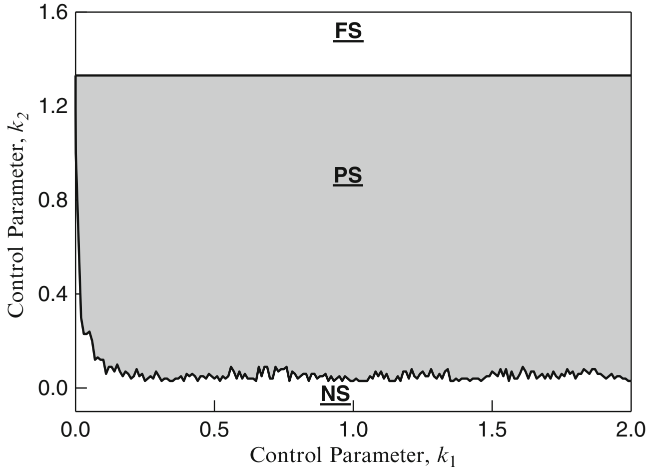

Fig. 5.6 Control parameter map of (k_1, k_2) for the sinusoidal synchronicity of the Duffing oscillator and the controlled pendulum (Duffing: $a_1 = a_2 = 1.0,\ d_1 = 0.25,\ A_0 = 0.4,\ \omega = 1.0$. Pendulum: $a_0 = 1.0,\ Q_0 = 0.275,\ \Omega = 2.18519$) (*FS* Full synchronization, *PS* Partial synchronization, *NS* Non-synchronization)

constraints. To view sinusoidal synchronization between the two distinct oscillators, the corresponding trajectories in phase plane are plotted in Fig. 5.7d. In phase plane, synchronization invariant domain is superimposed on phase plane. The synchronization invariant domains are shaded, and the rest region in phase space is the non-synchronization domain. To observe the existence of the synchronization for long time, the switching points of the two systems are presented for 10,000 periods of the master system in Fig. 5.7e, f. The black and red points are for appearance and vanishing of the synchronization.

From the parameter map, at $(k_1, k_2) = (1.0, 3.0)$, the fully sinusoidal chaotic synchronization between the controlled pendulum and the Duffing oscillator can be illustrated. As in Fig. 5.7, the time-histories of displacements, velocities, and G-functions plus trajectories in phase planes are illustrated in Fig. 5.8a–d. From the G-functions, the controlled pendulum and the Duffing oscillators are fully synchronized under the sinusoidal constraint. In Fig. 5.8d, the synchronization invariant domain with trajectories is also superimposed on phase space. The trajectories for the full sinusoidal synchronization of the pendulum and Duffing oscillators are in the synchronization invariant domain. The function synchronization cannot be observed intuitively. To determine whether two distinct dynamical systems are synchronized under specific function constraints or not, the G-function should be used. Otherwise, it is very difficult to determine the synchronization of two distinct dynamical systems. The synchronization theory presented in this book can provide a unique way to determine the synchronizations of the two distinct dynamical systems.

5.3.3 Sinusoidal Synchronizations of Periodic Motions

For period-1 motion synchronization, consider the following system parameters.

$$\begin{aligned} &a_1 = a_2 = 1.0, d_1 = 0.25, A_0 = 0.48, \omega = 1.0 \\ &a_0 = 1.0, Q_0 = 0.275, \Omega = 2.18519 \end{aligned} \tag{5.105}$$

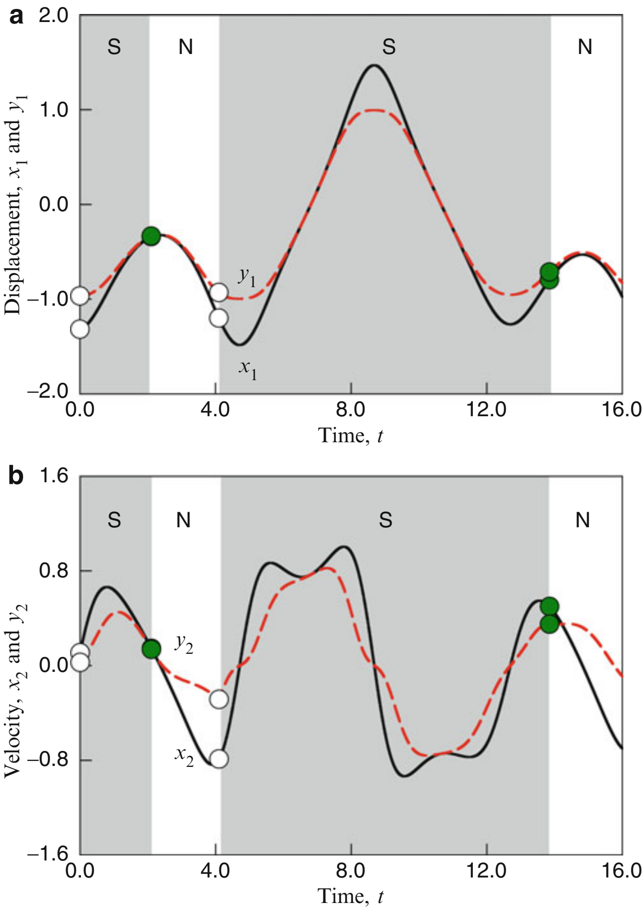

Fig. 5.7 Partial synchronization of the Duffing oscillator and the controlled pendulum: (**a**) displacement, (**b**) velocity, (**c**) G-function, (**d**) phase plane, (**e**) switching point for master, (**f**) switching points for slave. (Control parameters: $k_1 = 1$ and $k_2 = 0.8$. Duffing: $a_1 = a_2 = 1.0$, $d_1 = 0.25, A_0 = 0.4, \omega = 1.0$. Pendulum: $a_0 = 1.0, Q_0 = 0.275, \Omega = 2.18517$) (Initial conditions: $(x_1, x_2) = (-1.31892,\ 0.10879)$ and $(y_1, y_2) = (-0.96845,\ 0.02711)$) (*S* Synchronization, *N* Non-synchronization). *Hollow* and *filled circular symbols* are synchronization appearance ($\underline{A}$) and vanishing ($\underline{V}$), respectively

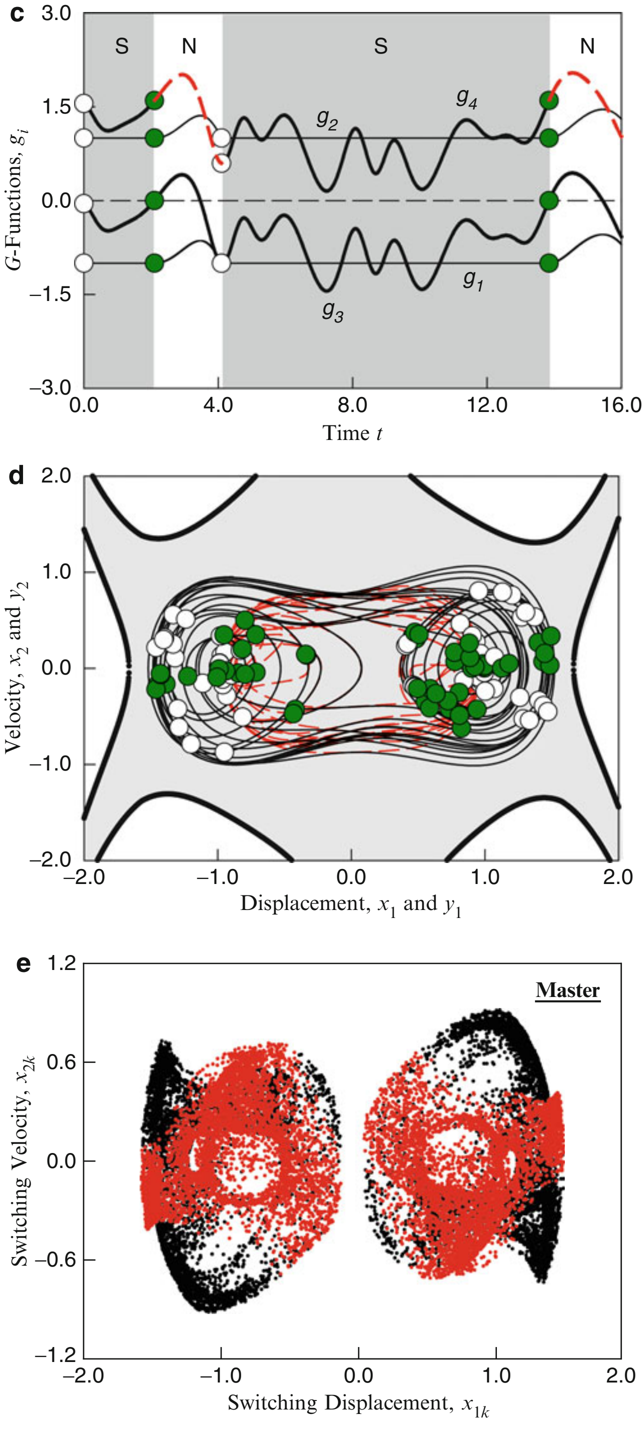

Fig. 5.7 (continued)

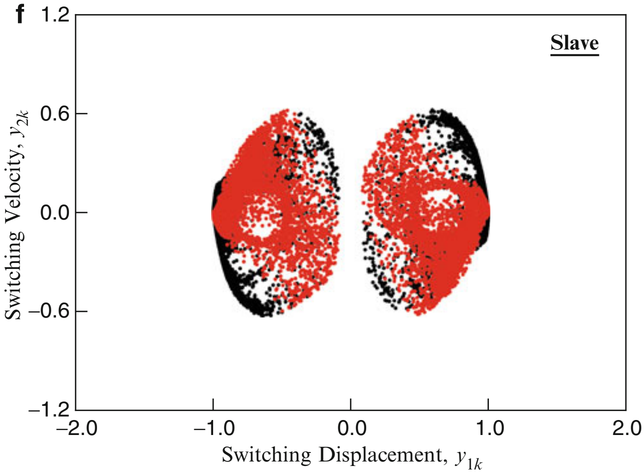

Fig. 5.7 (continued)

Consider the initial conditions $(x_1, x_2) = (0.510198,\ 1.32058)$ and $(y_1, y_2) = (0.48835,\ 1.15240)$, the Duffing oscillator exhibits period-1 motion, and the pendulum without control still possesses the chaotic motion. The constraint conditions ($y_1 = \sin x_1$ and $y_2 = x_2 \cos x_1$) are applied, the corresponding period-1 motion synchronization of the controlled pendulum with the Duffing oscillator will be presented herein. The sinusoidal period-1 motion synchronization scenario for the controlled pendulum and the Duffing oscillator is presented in Fig. 5.9 with $k_1 = 1$ through switching points versus control parameter k_2. $y_{1k} = \sin x_{1k}$ and $y_{2k} = x_{2k} \cos x_{1k}$. The switching displacement, switching velocity, and switching phases for synchronization of the controlled pendulum with the Duffing oscillator are illustrated in Fig. 5.9a–e, respectively. Compared to chaotic motion synchronization, the periodic motion synchronization between the two systems are very smooth and regular. From such period-1 motion synchronization scenario, the partial, sinusoidal, chaotic synchronization of the controlled pendulum with the Duffing oscillator is in the range of $k_2 \in (0.123,\ 1.798)$. If $k_2 \in (0, 0.123)$, no sinusoidal synchronization between the two systems can be obtained. If $k_2 \in (1.978, \infty)$, the full, sinusoidal synchronization of such two systems is achieved.

Consider the period-3 motion synchronization with the following parameters

$$\begin{aligned} &a_1 = a_2 = 1.0, d_1 = 0.25, A_0 = 0.33, \omega = 1.0 \\ &a_0 = 1.0, Q_0 = 0.275, \Omega = 2.18519. \end{aligned} \tag{5.106}$$

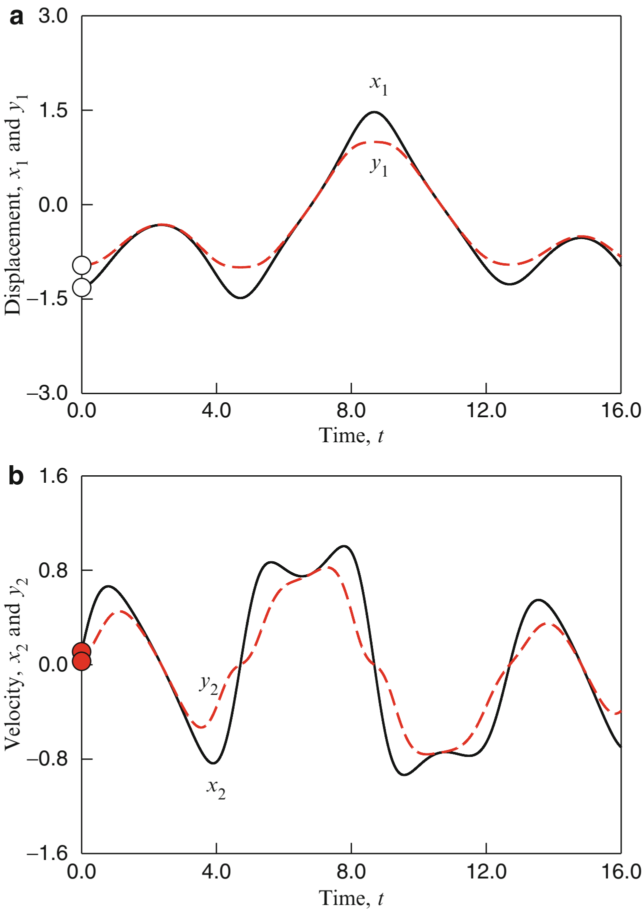

Fig. 5.8 Full synchronization of the Duffing oscillator and the controlled pendulum: (**a**) displacement, (**b**) velocity, (**c**) G-functions, (**d**) phase plane. (Control parameters: $k_1 = 1$ and $k_2 = 3$. Duffing: $a_1 = a_2 = 1.0,\ d_1 = 0.25,\ A_0 = 0.4,\ \omega = 1.0$. Pendulum: $a_0 = 1.0,\ Q_0 = 0.275,\ \Omega = 2.18519$) (Initial conditions: $(x_1, x_2) = (-1.31892,\ 0.10879)$ and $(y_1, y_2) = (-0.96845, 0.02711)$) (*FS* Full synchronization)

For $(x_1, x_2) = (0.472975,\ 0.440897)$ and $(y_1, y_2) = (0.455537,\ 0.392494)$, the Duffing oscillator exhibits period-3 motion, and the pendulum without control still possesses the chaotic motion. The sinusoidal period-3 motion synchronization scenario for the controlled pendulum and the Duffing oscillator will not be presented. However, the control parameter map $(k_1,\ k_2)$ is obtained and presented in Fig. 5.10a for the period-3 motion synchronization. For $k_2 > 1.182$ and $k_1 \neq 0$, the full sinusoidal synchronization of the two systems exists. For $k_1 < 1.76$, the bottom

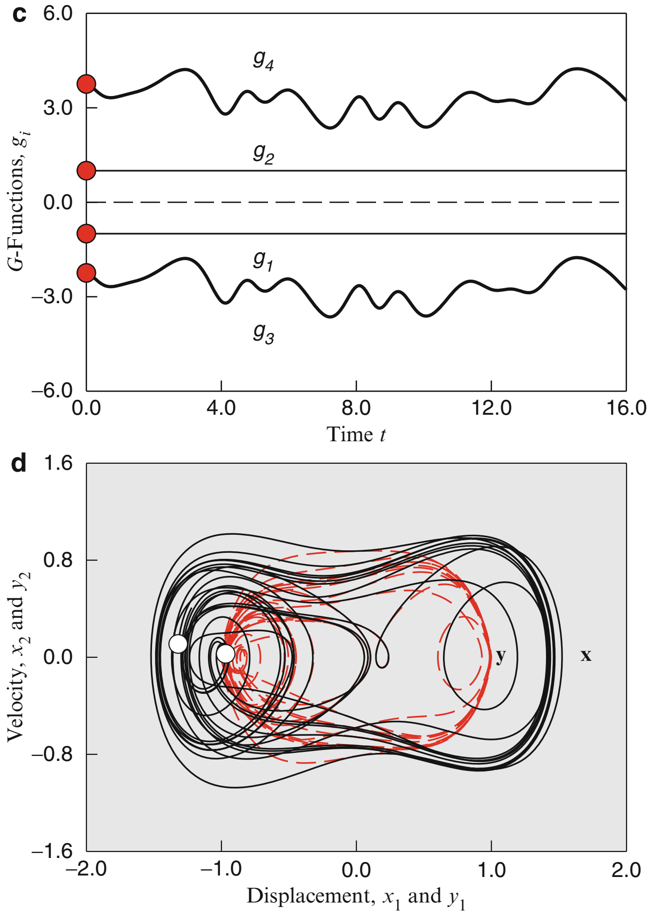

Fig. 5.8 (continued)

boundary of parameter map is zigzagged because the instantaneous synchronization exists. In addition, the control parameter map $(k_1,\ k_2)$ for period-1 motion synchronicity is obtained and presented in Fig. 5.10b. The shaded area is a partial sinusoidal synchronization zone. For $k_2 > 1.978$ and $k_1 \neq 0$, the sinusoidal synchronization of the two systems exists. The non-synchronization area is observed for small k_2. The bottom boundary of the partial synchronization in the control parameter map is much smooth.

For $(k_1,\ k_2) = (1.0,\ 1.2)$ in the parameter map, the partial sinusoidal, period-1 motion synchronization of the controlled pendulum and the Duffing oscillator is illustrated in Fig. 5.11. The time-histories of displacements, velocities, and

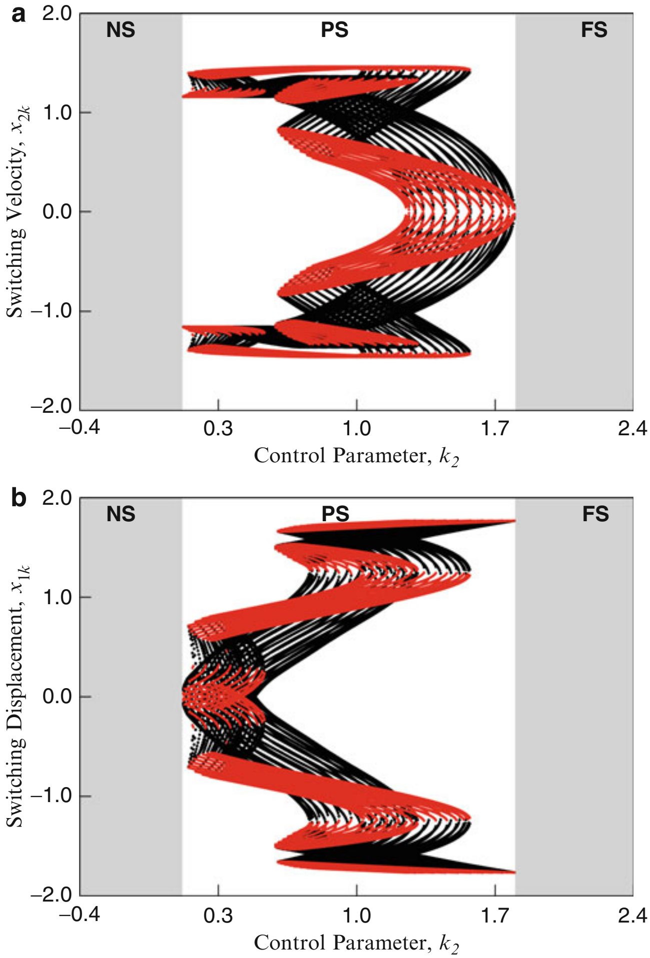

Fig. 5.9 Sinusoidal period-1 synchronization scenario of switching points versus control parameter k_2: (**a**) switching displacement and (**b**) switching velocity of master system; (**c**) switching displacement and (**d**) switching velocity of slave system; (**e**) switching phase (Control parameter: $k_1 = 1$. Duffing: $a_1 = a_2 = 1.0$, $d_1 = 0.25$, $A_0 = 0.48$, $\omega = 1.0$. Pendulum: $a_0 = 1.0$, $Q_0 = 0.275$, $\Omega = 2.18519$) (*FS* Full synchronization, *PS* Partial synchronization, *NS* Non-synchronization)

G-functions plus trajectories in phase planes are illustrated in Fig. 5.11a–d, respectively. The sinusoidal, period-1 motion synchronization and non-synchronization of the displacements and velocities for the two oscillators are presented in Fig. 5.11a, b. The shaded portions are also for synchronization, and the rest portions are for non-synchronization. In the two plots, the sinusoidal periodic motion synchronization

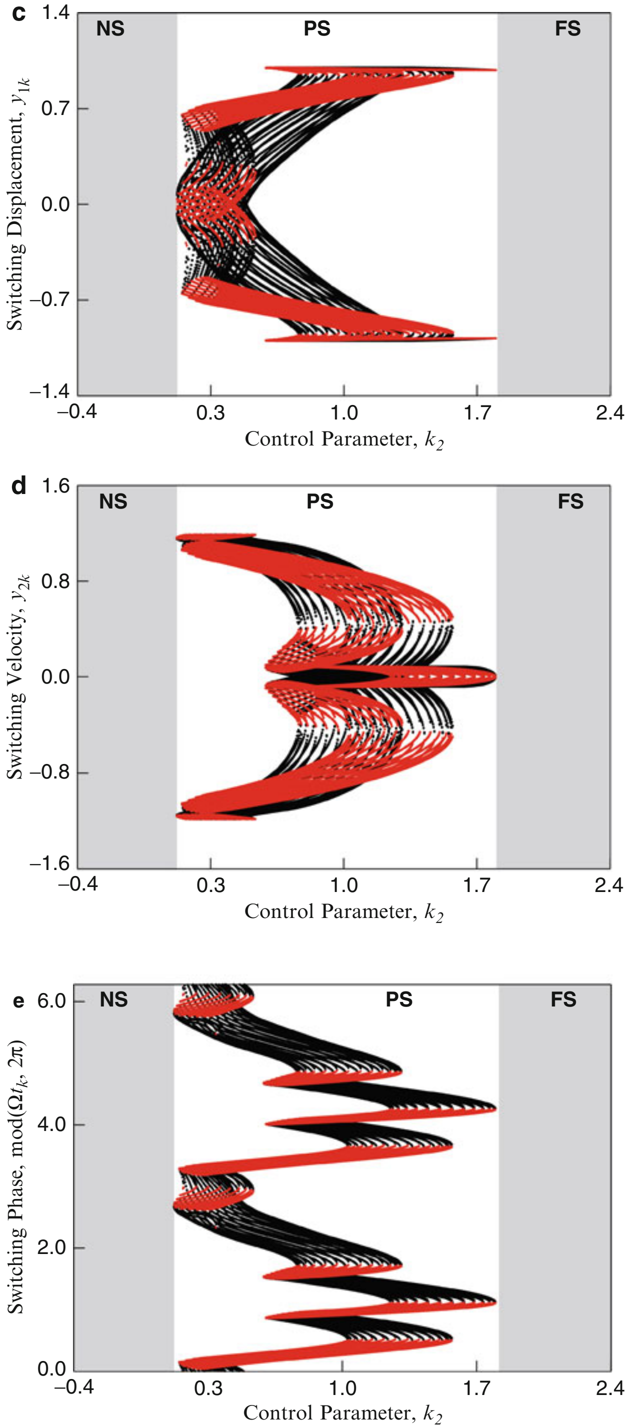

Fig. 5.9 (continued)

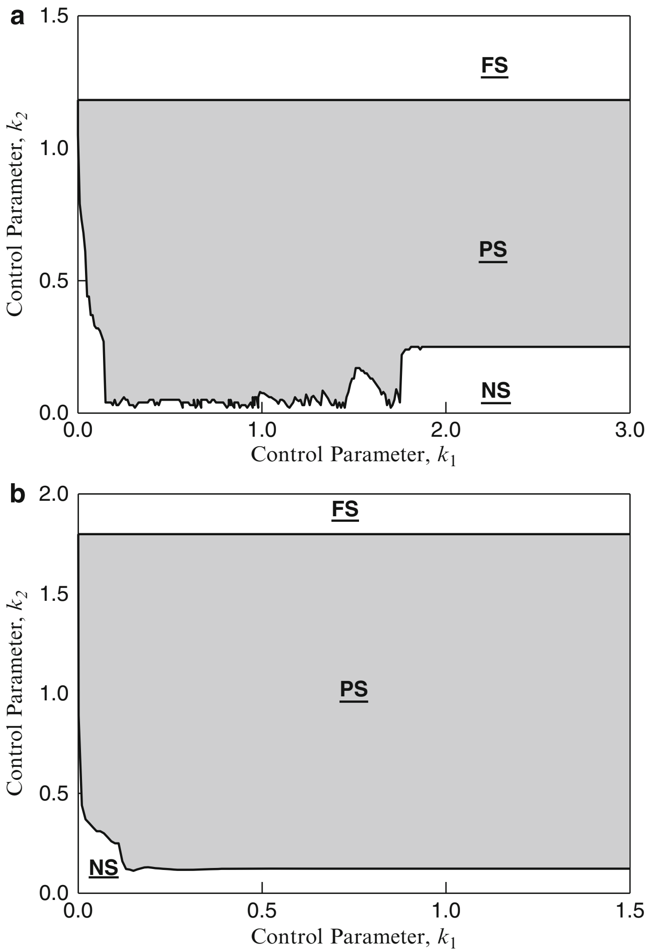

Fig. 5.10 Control parameter map of (k_1, k_2) for the sinusoidal periodic synchronicity of the Duffing oscillator and the controlled pendulum: (**a**) period-3 $(A_0 = 0.33)$ and (**b**) period-1 $(A_0 = 0.48)$ (Duffing: $a_1 = a_2 = 1.0,\ d_1 = 0.25,\ \omega = 1.0$. Pendulum: $a_0 = 1.0,\ Q_0 = 0.275,\ \Omega = 2.18519$) (*FS* Full synchronization, *PS* Partial synchronization, *NS* Non-synchronization)

cannot be intuitively observed. Thus, the time-histories of G-functions should be used to determine the synchronicity, as shown in Fig. 5.11c. The G-functions give the sinusoidal, period-1 motion synchronicity between the two oscillators under the sinusoidal constraint. The trajectories of the master and slave systems in phase plane are plotted in Fig. 5.11d. In phase plane, synchronization invariant domain is superimposed on phase plane. The synchronization invariant domains are shaded,

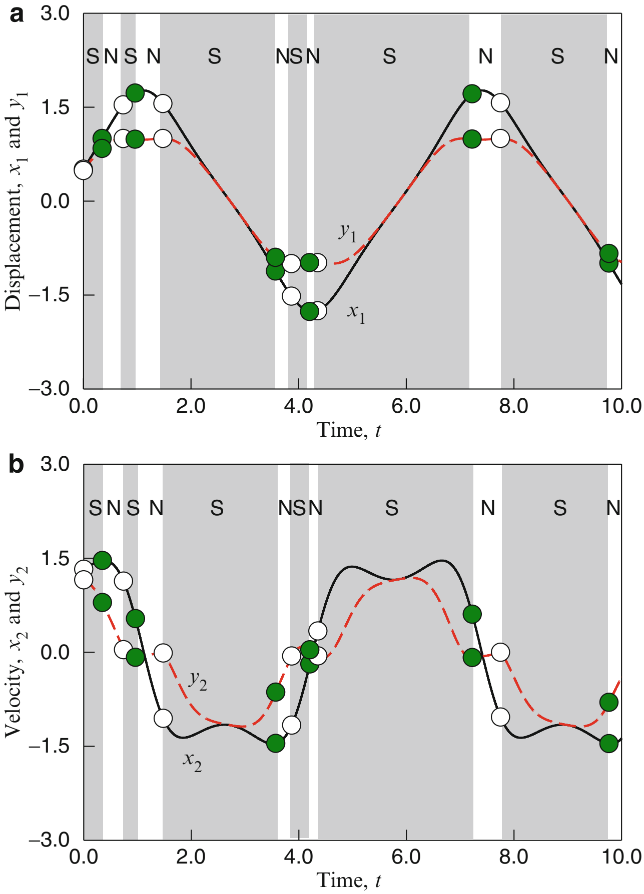

Fig. 5.11 Partial synchronization of the Duffing oscillator and the controlled pendulum: (**a**) displacement, (**b**) velocity, (**c**) G-function, and (**d**) phase plane. (Control parameters: $k_1 = 1$ and $k_2 = 1.2$. Duffing: $a_1 = a_2 = 1.0$, $d_1 = 0.25$, $A_0 = 0.48$, $\omega = 1.0$. Pendulum: a$a_0 = 1.0$, $Q_0 = 0.275$, $\Omega = 2.18517$). (Initial conditions: $(x_1, x_2) = (0.510198,\ 1.32058)$ and $(y_1, y_2) = (0.48835,\ 1.1524)$) (*S* Synchronization; *N* Non-synchronization). *Hollow* and *filled circular symbols* are synchronization appearance (A) and vanishing (V), respectively. The *shaded area* in phase plane is the synchronization invariant domain

and the rest region is the non-synchronization domain. The vanishing and appearing points of the master system are near the boundary.

For $(k_1, k_2) = (1.0, 3.0)$ in the parameter map, the fully sinusoidal period-1 motion synchronization between the controlled pendulum and the Duffing oscillator is illustrated in Fig. 5.12. The time-histories of displacements, velocities, and G-functions plus trajectories in phase planes are shown in Fig. 5.12a–d. From the G-functions, the controlled pendulum and the Duffing oscillators

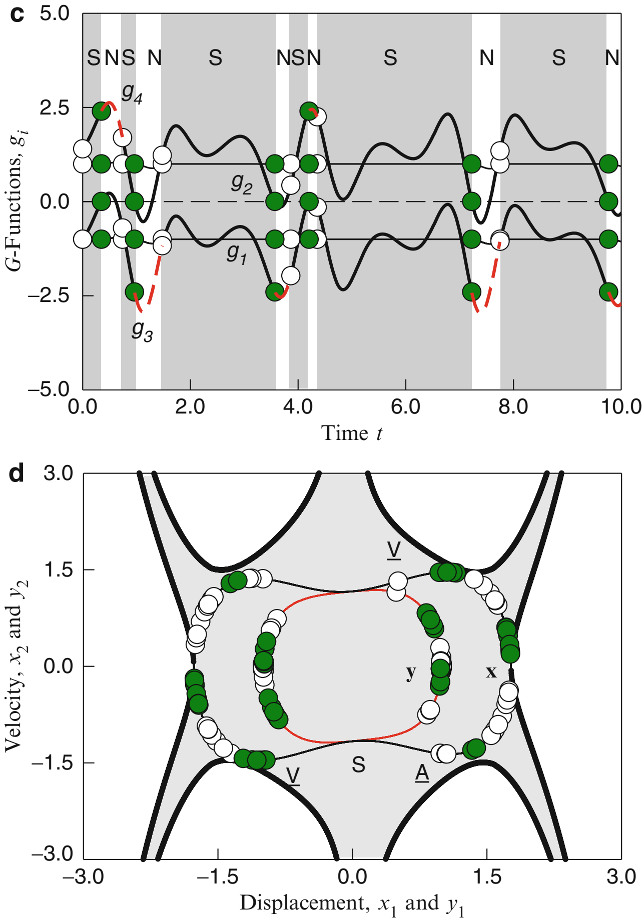

Fig. 5.11 (continued)

are fully synchronized for period-1 motion under the sinusoidal constraint. In Fig. 5.12d, the synchronization invariant domain with trajectories is superimposed on phase space. The trajectories for the full sinusoidal period-1 motion synchronization of the controlled pendulum and Duffing oscillators are in such an invariant domain.

For period-3 motion synchronization between the controlled pendulum and the Duffing oscillator, only trajectories in phase plane are presented in Fig. 5.13a, b at $k_1 = 1.0$ for partial $(k_2 = 0.6)$ and full $(k_2 = 2.0)$ synchronizations with the corresponding synchronization invariant domains, respectively. Such illustrations can help one understand the synchronization dynamics of the controlled pendulum

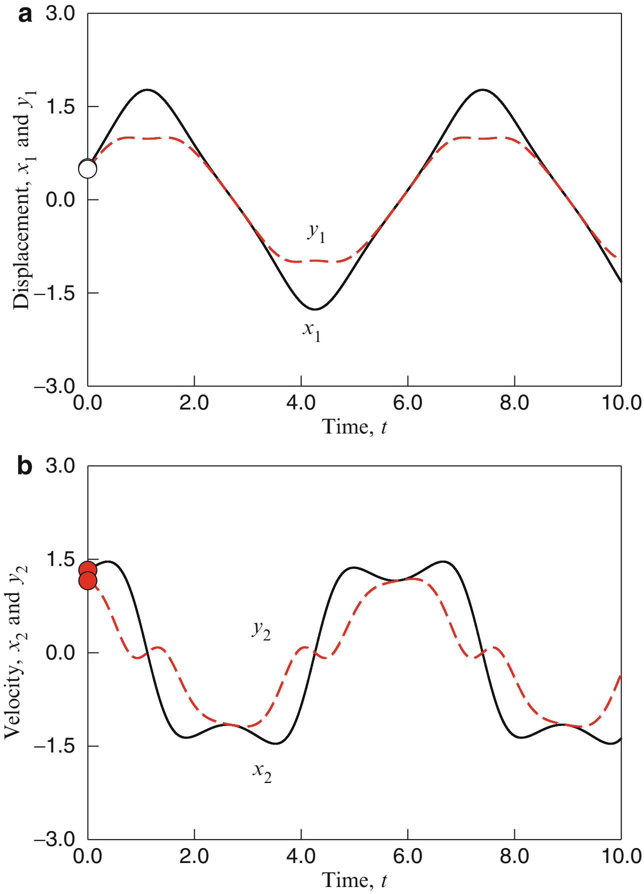

Fig. 5.12 Full sinusoidal periodic synchronization of the Duffing oscillator and the controlled pendulum: (**a**) displacement, (**b**) velocity, (**c**) G-function, (**d**) phase plane. (Control parameters: $k_1 = 1$ and $k_2 = 3$. Duffing: $a_1 = a_2 = 1.0$, $d_1 = 0.25$, $A_0 = 0.48$, $\omega = 1.0$. Pendulum: $a_0 = 1.0$, $Q_0 = 0.275$, $\Omega = 2.18517$). (Initial conditions: $(x_1, x_2) = (0.510198,\ 1.32058)$ and $(y_1, y_2) = (0.48835,\ 1.1524)$). The *shaded area* in phase plane is the synchronization invariant domain

and the Duffing oscillator. Similarly, other periodic motions synchronization can be investigated under such sinusoidal constraints. If the chaotic pendulum is as a master system, the controlled Duffing oscillator synchronizing with the pendulum under sinusoidal constraints was presented in Min and Luo [7]. The methodology presented herein can be used for secure communications and cryptography.

Luo and Min [8–11] used the synchronization theory of two dynamical systems in Luo [2] to discuss the identical synchronizations of the different dynamical

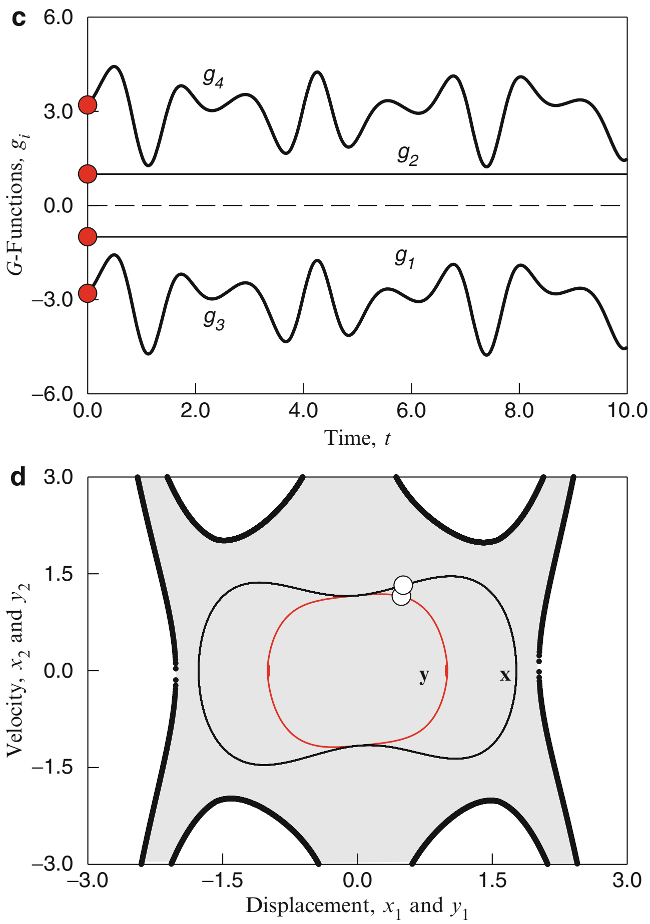

Fig. 5.12 (continued)

systems. The synchronization of two distinct dynamical systems was determined without the Lyapunov method. The periodic and chaotic synchronizations of two distinct dynamical systems were presented. The invariant domain of synchronization was discovered, which can help one easily determine the two dynamical systems synchronization. In addition, Min and Luo [12] used the new theory of dynamical system synchronization to investigate the noised gyroscope systems synchronizing with the expected gyroscope systems. The partial synchronization is an important phenomenon to be observed. Such results on gyroscope systems provide a very good example for engineering application. This synchronization theory can be easily applied for maneuvering targets tracking and space vehicles tracking and connections.

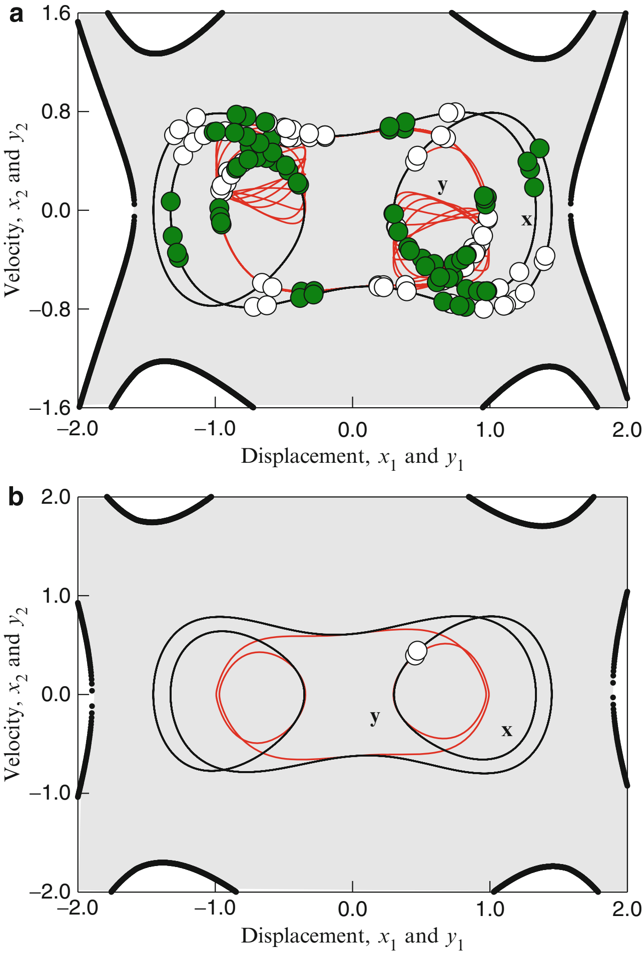

Fig. 5.13 Phase planes for sinusoidal period-3 synchronization of the Duffing oscillator and the controlled pendulum: (**a**) partial synchronization ($k_2 = 0.6$), (**b**) full synchronization ($k_2 = 2.0$). (Control parameter: $k_1 = 1$, Duffing: $a_1 = a_2 = 1.0$, $d_1 = 0.25$, $A_0 = 0.33$, $\omega = 1.0$. Pendulum: $a_0 = 1.0$, $Q_0 = 0.275$, $\Omega = 2.18517$). (Initial conditions: $(x_1, x_2) = (0.472975,\ 0.440897)$ and $(y_1, y_2) = (0.455537,\ 0.392494)$). The *shaded area* is the synchronization invariant domain

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Chapter 6
Discrete Systems Synchronization

As in Luo [1, 2], a set of concepts on "Ying" and "Yang" in discrete dynamical systems will be presented. Based on the Ying-Yang theory, the complete dynamics of discrete dynamical systems will be presented for an understanding of dynamical behaviors. From the ideas of the Ying-Yang theory of discrete dynamical systems, the companion and synchronization of discrete dynamical systems will be presented herein, and the corresponding conditions will be presented as an integrity part of dynamical system synchronization. The synchronization dynamics of Duffing and Henon maps will be discussed.

6.1 Discrete Systems with a Single Nonlinear Map

Definition 6.1 Consider an implicit vector function $\mathbf{f}: D \to D$ on an open set $D \subset \mathscr{R}^n$ in an n-dimensional discrete dynamical system. For $\mathbf{x}_k, \mathbf{x}_{k+1} \in D$, there is a discrete relation as

$$\mathbf{f}(\mathbf{x}_k, \mathbf{x}_{k+1}, \mathbf{p}) = \mathbf{0}, \tag{6.1}$$

where the vector function is $\mathbf{f} = (f_1, f_2, \cdots, f_n)^{\mathrm{T}} \in \mathscr{R}^n$ and discrete variable vector is $\mathbf{x}_k = (x_{k1}, x_{k2}, \cdots, x_{kn})^{\mathrm{T}} \in D$ with a parameter vector $\mathbf{p} = (p_1, p_2, \cdots, p_m)^{\mathrm{T}} \in \mathscr{R}^m$.

Definition 6.2 For a discrete dynamical system in Eq. (6.1), the positive and negative discrete sets are defined by

$$\left.\begin{aligned} \Sigma_+ &= \{\mathbf{x}_{k+i} | \mathbf{x}_{k+i} \in \mathscr{R}^n, i \in \mathbb{Z}_+\} \subset D \text{ and} \\ \Sigma_- &= \{\mathbf{x}_{k-i} | \mathbf{x}_{k-i} \in \mathscr{R}^n, i \in \mathbb{Z}_+\} \subset D, \end{aligned}\right\} \tag{6.2}$$

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respectively. The discrete set is

$$\Sigma = \Sigma_{+} \cup \Sigma_{-}. \tag{6.3}$$

A positive mapping is defined as

$$P_{+} : \Sigma \rightarrow \Sigma_{+} \Rightarrow P_{+} : \mathbf{x}_k \rightarrow \mathbf{x}_{k+1} \tag{6.4}$$

and a negative mapping is defined by

$$P_{-} : \Sigma \rightarrow \Sigma_{-} \Rightarrow P_{-} : \mathbf{x}_k \rightarrow \mathbf{x}_{k-1}. \tag{6.5}$$

Definition 6.3 For a discrete dynamical system in Eq. (6.1), consider two points $\mathbf{x}_k \in D$ and $\mathbf{x}_{k+1} \in D$, and there is a specific, differentiable, vector function $\mathbf{g} \in \mathscr{R}^n$ to make $\mathbf{g}(\mathbf{x}_k, \mathbf{x}_{k+1}, \boldsymbol{\lambda}) = \mathbf{0}$.

(i) The stable solution based on $\mathbf{x}_{k+1} = P_{+}\mathbf{x}_k$ for the positive mapping P_{+} is called the "Yang" of the discrete dynamical system in Eq. (6.1) in sense of $\mathbf{g}(\mathbf{x}_k, \mathbf{x}_{k+1}, \boldsymbol{\lambda}) = \mathbf{0}$ if $\mathbf{f}(\mathbf{x}_k, \mathbf{x}_{k+1}, \mathbf{p}) = \mathbf{0}$ with $\mathbf{g}(\mathbf{x}_k, \mathbf{x}_{k+1}, \boldsymbol{\lambda}) = \mathbf{0}$ have the P_{+}-1 solutions $(\mathbf{x}_k^*, \mathbf{x}_{k+1}^*)$.
(ii) The stable solution based on $\mathbf{x}_k = P_{-}\mathbf{x}_{k+1}$ for the negative mapping P_{-} is called the "Ying" of the discrete dynamical system in Eq. (6.1) in sense of $\mathbf{g}(\mathbf{x}_k, \mathbf{x}_{k+1}, \boldsymbol{\lambda}) = \mathbf{0}$ if $\mathbf{f}(\mathbf{x}_k, \mathbf{x}_{k+1}, \mathbf{p}) = \mathbf{0}$ with $\mathbf{g}(\mathbf{x}_k, \mathbf{x}_{k+1}, \boldsymbol{\lambda}) = \mathbf{0}$ have the P_{-}-1 solutions $(\mathbf{x}_k^*, \mathbf{x}_{k+1}^*)$.
(iii) The solution based on $\mathbf{x}_{k+1} = P_{+}\mathbf{x}_k$ is called the "Ying-Yang" for the positive mapping P_{+} of the discrete dynamical system in Eq. (6.1) in sense of $\mathbf{g}(\mathbf{x}_k, \mathbf{x}_{k+1}, \boldsymbol{\lambda}) = \mathbf{0}$ if $\mathbf{f}(\mathbf{x}_k, \mathbf{x}_{k+1}, \mathbf{p}) = \mathbf{0}$ with $\mathbf{g}(\mathbf{x}_k, \mathbf{x}_{k+1}, \boldsymbol{\lambda}) = \mathbf{0}$ have the P_{+}-1 solutions $(\mathbf{x}_k^*, \mathbf{x}_{k+1}^*)$ and the eigenvalues of $DP_{+}(\mathbf{x}_k^*)$ are distributed inside and outside the unit cycle.
(iv) The solution based on $\mathbf{x}_k = P_{-}\mathbf{x}_{k+1}$ is called the "Ying-Yang" for the negative mapping P_{-} of the discrete dynamical system in Eq. (6.1) in sense of $\mathbf{g}(\mathbf{x}_k, \mathbf{x}_{k+1}, \boldsymbol{\lambda}) = \mathbf{0}$ if $\mathbf{f}(\mathbf{x}_k, \mathbf{x}_{k+1}, \mathbf{p}) = \mathbf{0}$ with $\mathbf{g}(\mathbf{x}_k, \mathbf{x}_{k+1}, \boldsymbol{\lambda}) = \mathbf{0}$ have the P_{-}-1 solutions $(\mathbf{x}_k^*, \mathbf{x}_{k+1}^*)$ and the eigenvalues of $DP_{-}(\mathbf{x}_{k+1}^*)$ are distributed inside and outside unit cycle.

Consider the positive and negative mappings are

$$\mathbf{x}_{k+1} = P_{+}\mathbf{x}_k \text{ and } \mathbf{x}_k = P_{-}\mathbf{x}_{k+1}. \tag{6.6}$$

For the simplest case, consider the constraint condition of $\mathbf{g}(\mathbf{x}_k, \mathbf{x}_{k+1}, \boldsymbol{\lambda}) = \mathbf{x}_{k+1} - \mathbf{x}_k = \mathbf{0}$. Thus, the positive and negative mappings have, respectively, the constraints

$$\mathbf{x}_{k+1} = \mathbf{x}_k \text{ and } \mathbf{x}_k = \mathbf{x}_{k+1}. \tag{6.7}$$

Both positive and negative mappings are governed by the discrete relation in Eq. (6.1). In other words, Eq. (6.6) gives

$$\mathbf{f}(\mathbf{x}_k, \mathbf{x}_{k+1}, \mathbf{p}) = \mathbf{0} \text{ and } \mathbf{f}(\mathbf{x}_k, \mathbf{x}_{k+1}, \mathbf{p}) = \mathbf{0}. \tag{6.8}$$

Setting the period-1 solution $\mathbf{x}_k^*$ and substitution of Eq. (6.7) into Eq. (6.8) gives

$$\mathbf{f}(\mathbf{x}_k^*, \mathbf{x}_k^*, \mathbf{p}) = \mathbf{0} \text{ and } \mathbf{f}(\mathbf{x}_k^*, \mathbf{x}_k^*, \mathbf{p}) = \mathbf{0}. \tag{6.9}$$

From the foregoing equation, the period-1 solutions for the positive and negative mappings are identical. The two relations for positive and negative mappings are illustrated in Fig. 6.1a, b, respectively. To determine the period-1 solution, the fixed points of Eq. (6.7) exist under constraints in Eq. (6.8), as also shown in Fig. 6.1. The two thick lines on the axis are two sets for the mappings from the starting to final states. The relation in Eq. (6.7) is presented by a solid curve. The intersection points of the curves and straight lines for relations in Eqs. (6.7) and (6.8) give the fixed points of Eq. (6.9), which are period-1 solutions, labeled by the circular symbols. However, their stability and bifurcation for the period-1 solutions are different. To determine the stability and bifurcation of the period-1 solution of the positive and negative mappings, the following theorem is stated.

Theorem 6.1 *For a discrete dynamical system in Eq. (6.1), there are two points* $\mathbf{x}_k \in D$ *and* $\mathbf{x}_{k+1} \in D$, *and two positive and negative mappings are*

$$\mathbf{x}_{k+1} = P_+\mathbf{x}_k \text{ and } \mathbf{x}_k = P_-\mathbf{x}_{k+1} \tag{6.10}$$

with

$$\mathbf{f}(\mathbf{x}_k, \mathbf{x}_{k+1}, \mathbf{p}) = \mathbf{0} \text{ and } \mathbf{f}(\mathbf{x}_k, \mathbf{x}_{k+1}, \mathbf{p}) = \mathbf{0}. \tag{6.11}$$

Suppose a specific, differentiable, vector function $\mathbf{g} \in \mathscr{R}^n$ *makes* $\mathbf{g}(\mathbf{x}_k, \mathbf{x}_{k+1}, \boldsymbol{\lambda}) = \mathbf{0}$ *hold. If the solutions* $(\mathbf{x}_k^*, \mathbf{x}_{k+1}^*)$ *of both* $\mathbf{f}(\mathbf{x}_k, \mathbf{x}_{k+1}, \mathbf{p}) = \mathbf{0}$ *and* $\mathbf{g}(\mathbf{x}_k, \mathbf{x}_{k+1}, \boldsymbol{\lambda}) = \mathbf{0}$ *exist, then the following conclusions in the sense of* $\mathbf{g}(\mathbf{x}_k, \mathbf{x}_{k+1}, \boldsymbol{\lambda}) = \mathbf{0}$ *hold.*

(i) *The stable* P_+-1 *solutions are the unstable* P_--1 *solutions with all eigenvalues of* $DP_-(\mathbf{x}_k^*)$ *outside the unit cycle, vice versa.*
(ii) *The unstable* P_+-1 *solutions with all eigenvalues of* $DP_+(\mathbf{x}_k^*)$ *outside the unit cycle are the stable* P_--1 *solutions, vice versa.*
(iii) *For the unstable* P_+-1 *solutions with eigenvalue distribution of* $DP_+(\mathbf{x}_k^*)$ *inside and outside the unit cycle, the corresponding* P_--1 *solution is also unstable with switching the eigenvalue distribution of* $DP_-(\mathbf{x}_k^*)$ *inside and outside the unit cycle, vice versa.*
(iv) *All the bifurcations of the stable and unstable* P_+-1 *solutions are all the bifurcations of the unstable and stable* P_--1 *solutions, respectively.*

Proof The proof can be referred to Luo [2] (Fig. 6.1). □

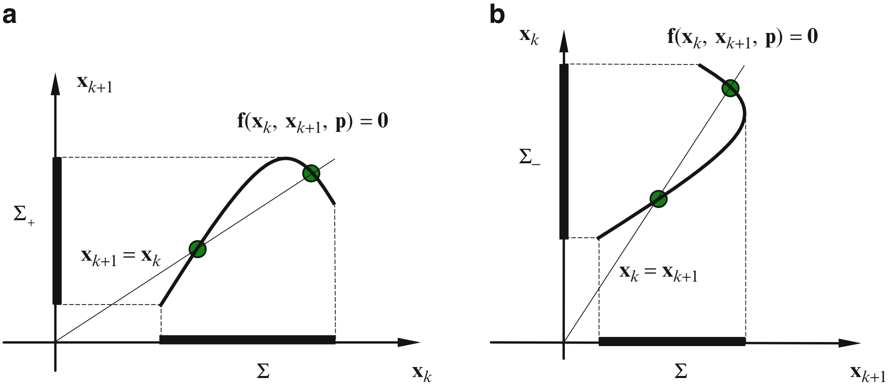

Fig. 6.1 Period-1 solution for (**a**) positive mapping and (**b**) negative mapping. The two *thick lines* on the axis are two sets for the mappings from the starting to final states. The mapping relation is presented by a *solid curve*. The *circular symbols* give period-1 solutions for the positive and negative mappings

From the foregoing theorem, the *Ying*, *Yang* and *Ying*-*Yang* states in discrete dynamical systems exist. To generate the above ideas to $P_+^{(N)}$-1 and $P_-^{(N)}$-1 solutions in discrete dynamical systems in sense of $\mathbf{g}(\mathbf{x}_k, \mathbf{x}_{k+N}, \boldsymbol{\lambda}) = \mathbf{0}$, the mapping structure consisting of N-positive or negative mappings is considered.

Definition 6.4 For a discrete dynamical system in Eq. (6.1), the mapping structures of N-mappings for the positive and negative mappings are defined as

$$\mathbf{x}_{k+N} = \underbrace{P_+ \circ P_+ \circ \cdots \circ P_+}_{N} \mathbf{x}_k = P_+^{(N)}\mathbf{x}_k, \tag{6.12}$$

$$\mathbf{x}_k = \underbrace{P_- \circ P_- \circ \cdots \circ P_-}_{N} \mathbf{x}_{k+N} = P_-^{(N)}\mathbf{x}_{k+N} \tag{6.13}$$

with

$$\mathbf{f}(\mathbf{x}_{k+i-1}, \mathbf{x}_{k+i}, \mathbf{p}) = \mathbf{0} \text{ for } i = 1, 2, \cdots, N \tag{6.14}$$

where $P_+^{(0)} = 1$ and $P_-^{(0)} = 1$ for $N = 0$.

Definition 6.5 For a discrete dynamical system in Eq. (6.1), consider two points $\mathbf{x}_{k+i-1} \in D$ $(i = 1, 2, \cdots, N)$ and $\mathbf{x}_{k+N} \in D$, and there is a specific, differentiable, vector function $\mathbf{g} \in \mathscr{R}^n$ to make $\mathbf{g}(\mathbf{x}_k, \mathbf{x}_{k+N}, \boldsymbol{\lambda}) = \mathbf{0}$.

(i) The stable solution based on $\mathbf{x}_{k+N} = P_+^{(N)}\mathbf{x}_k$ for the positive mapping P_+ is called the "Yang" of the discrete dynamical system in Eq. (6.1) in sense *of*

$\mathbf{g}(\mathbf{x}_k, \mathbf{x}_{k+N}, \boldsymbol{\lambda}) = \mathbf{0}$ if the solutions $(\mathbf{x}_k^*, \mathbf{x}_{k+1}^*, \cdots, \mathbf{x}_{k+N}^*)$ of Eq. (6.14) with $\mathbf{g}(\mathbf{x}_k, \mathbf{x}_{k+N}, \boldsymbol{\lambda}) = \mathbf{0}$ exist.

(ii) The stable solution based on $\mathbf{x}_k = P_-^{(N)}\mathbf{x}_{k+N}$ for the negative mapping P_- is called the "Ying" of the discrete dynamical system in Eq. (6.1) in sense of $\mathbf{g}(\mathbf{x}_k, \mathbf{x}_{k+N}, \boldsymbol{\lambda}) = \mathbf{0}$ if the solutions $(\mathbf{x}_k^*, \mathbf{x}_{k+1}^*, \cdots, \mathbf{x}_{k+N}^*)$ of Eq. (6.14) with $\mathbf{g}(\mathbf{x}_k, \mathbf{x}_{k+N}, \boldsymbol{\lambda}) = \mathbf{0}$ exist.

(iii) The solution based on $\mathbf{x}_{k+N} = P_+^{(N)}\mathbf{x}_k$ is called the "Ying-Yang" for the positive mapping P_+ of the discrete dynamical system in Eq. (6.1) in sense of $\mathbf{g}(\mathbf{x}_k, \mathbf{x}_{k+N}, \boldsymbol{\lambda}) = \mathbf{0}$ if the solutions $(\mathbf{x}_k^*, \mathbf{x}_{k+1}^*, \cdots, \mathbf{x}_{k+N}^*)$ of Eq. (6.14) with $\mathbf{g}(\mathbf{x}_k, \mathbf{x}_{k+N}, \boldsymbol{\lambda}) = \mathbf{0}$ exist and the eigenvalues of $DP_+^{(N)}(\mathbf{x}_k^*)$ are distributed inside and outside the unit cycle.

(iv) The solution based on $\mathbf{x}_k = P_-^{(N)}\mathbf{x}_{k+N}$ is called the "Ying-Yang" for the negative mapping P_- of the discrete dynamical system in Eq. (6.1) in sense *of* $\mathbf{g}(\mathbf{x}_k, \mathbf{x}_{k+N}, \boldsymbol{\lambda}) = \mathbf{0}$ if the solutions $(\mathbf{x}_k^*, \mathbf{x}_{k+1}^*, \cdots, \mathbf{x}_{k+N}^*)$ of Eq. (6.14) with $\mathbf{g}(\mathbf{x}_k, \mathbf{x}_{k+N}, \boldsymbol{\lambda}) = \mathbf{0}$ exist and the eigenvalues of $DP_-^{(N)}(\mathbf{x}_{k+N}^*)$ are distributed inside and outside unit cycle.

To determine the Ying-Yang properties of $P_+^{(N)}$-1 and $P_-^{(N)}$-1 in the discrete mapping system in Eq. (6.1), the corresponding theorem is presented as follows.

Theorem 6.2 *For a discrete dynamical system in Eq. (6.1), there are two points* $\mathbf{x}_k \in D$ *and* $\mathbf{x}_{k+N} \in D$, *and two positive and negative mappings are*

$$\mathbf{x}_{k+N} = P_+^{(N)}\mathbf{x}_k \text{ and } \mathbf{x}_k = P_-^{(N)}\mathbf{x}_{k+N}, \tag{6.15}$$

and $\mathbf{x}_{k+i} = P_+\mathbf{x}_{k+i-1}$ *and* $\mathbf{x}_{k+i-1} = P_-\mathbf{x}_{k+i}$ *can be governed by*

$$\mathbf{f}(\mathbf{x}_{k+i-1}, \mathbf{x}_{k+i}, \mathbf{p}) = \mathbf{0} \text{ for } i = 1, 2, \cdots, N. \tag{6.16}$$

Suppose a specific, differentiable, vector function of $\mathbf{g} \in \mathscr{R}^n$ *makes* $\mathbf{g}(\mathbf{x}_k, \mathbf{x}_{k+N}, \boldsymbol{\lambda}) = 0$ *hold. If the solutions* $(\mathbf{x}_k^*, \cdots, \mathbf{x}_{k+i}^*)$ *of Eq. (6.16) with* $\mathbf{g}(\mathbf{x}_k, \mathbf{x}_{k+N}, \boldsymbol{\lambda}) = \mathbf{0}$ *exist, then the following conclusions in the sense of* $\mathbf{g}(\mathbf{x}_k, \mathbf{x}_{k+N}, \boldsymbol{\lambda}) = \mathbf{0}$ *hold.*

(i) *The stable* $P_+^{(N)}$-1 *solution is the unstable* $P_-^{(N)}$-1 *solution with all eigenvalues of* $DP_-^{(N)}(\mathbf{x}_{k+N}^*)$ *outside the unit cycle, vice versa.*

(ii) *The unstable* $P_+^{(N)}$-1 *solution with all eigenvalues of* $DP_+^{(N)}(\mathbf{x}_k^*)$ *outside the unit cycle is the stable* $P_-^{(N)}$-1 *solution, vice versa.*

(iii) *For the unstable* $P_+^{(N)}$-1 *solution with eigenvalue distribution of* $DP_+^{(N)}(\mathbf{x}_k^*)$ *inside and outside the unit cycle, the corresponding* $P_-^{(N)}$-1 *solution is also unstable with switching eigenvalue distribution of* $DP_-^{(N)}(\mathbf{x}_{k+N}^*)$ *inside and outside the unit cycle, vice versa.*

(iv) *All the bifurcations of the stable and unstable* $P_+^{(N)}$-1 *solution are all the bifurcations of the unstable and stable* $P_-^{(N)}$-1 *solution, respectively.*

Proof The proof can be referred to Luo [2]. □

Theorem 6.3 *For a discrete dynamical system in Eq. (6.1), there are two points* $\mathbf{x}_k \in D$ *and* $\mathbf{x}_{k+N} \in D$. *If the period-doubling cascade of the* $P_+^{(N)}$-1 *and* $P_-^{(N)}$-1 *solution occurs, the corresponding mapping structures are given by*

$$\begin{aligned}
&\mathbf{x}_{k+2N} = P_+^{(N)} \circ P_+^{(N)}\mathbf{x}_k = P_+^{(2N)}\mathbf{x}_k \text{ and } \mathbf{g}(\mathbf{x}_k, \mathbf{x}_{k+2N}, \boldsymbol{\lambda}) = \mathbf{0};\\
&\mathbf{x}_{k+2^2N} = P_+^{(2N)} \circ P_+^{(2N)}\mathbf{x}_k = P_+^{(2^2N)}\mathbf{x}_k \text{ and } \mathbf{g}(\mathbf{x}_k, \mathbf{x}_{k+2^2N}, \boldsymbol{\lambda}) = \mathbf{0};\\
&\vdots\\
&\mathbf{x}_{k+2^lN} = P_+^{(2^{l-1}N)} \circ P_+^{(2^{l-1}N)}\mathbf{x}_k = P_+^{(2^lN)}\mathbf{x}_k \text{ and } \mathbf{g}(\mathbf{x}_k, \mathbf{x}_{k+2^lN}, \boldsymbol{\lambda}) = \mathbf{0};
\end{aligned} \tag{6.17}$$

for positive mappings and

$$\begin{aligned}
&\mathbf{x}_k = P_-^{(N)} \circ P_-^{(N)}\mathbf{x}_{k+2N} = P_-^{(2N)}\mathbf{x}_{k+2N} \text{ and } \mathbf{g}(\mathbf{x}_k, \mathbf{x}_{k+2N}, \boldsymbol{\lambda}) = \mathbf{0};\\
&\mathbf{x}_k = P_-^{(2N)} \circ P_-^{(2N)}\mathbf{x}_{k+2^2N} = P_-^{(2^2N)}\mathbf{x}_{k+2^2N} \text{ and } \mathbf{g}(\mathbf{x}_k, \mathbf{x}_{k+2^2N}, \boldsymbol{\lambda}) = \mathbf{0};\\
&\vdots\\
&\mathbf{x}_k = P_-^{(2^{l-1}N)} \circ P_-^{(2^{l-1}N)}\mathbf{x}_{k+2^lN} = P_-^{(2^lN)}\mathbf{x}_{k+2^lN} \text{ and } \mathbf{g}(\mathbf{x}_k, \mathbf{x}_{k+2^lN}, \boldsymbol{\lambda}) = \mathbf{0}
\end{aligned} \tag{6.18}$$

for negative mapping, then the following statements hold, i.e.,

(i) *The stable chaos generated by the limit state of the stable* $P_+^{(2^lN)}$-1 *solutions* ($l \to \infty$) *in sense of* $\mathbf{g}(\mathbf{x}_k, \mathbf{x}_{k+2^lN}, \boldsymbol{\lambda}) = \mathbf{0}$ *is the unstable chaos generated by the limit state of the unstable stable* $P_-^{(2^lN)}$-1 *solution* ($l \to \infty$) *in sense of* $\mathbf{g}(\mathbf{x}_k, \mathbf{x}_{k+2^lN}, \boldsymbol{\lambda}) = \mathbf{0}$ *with all eigenvalue distribution of* $DP_-^{(2^lN)}$ *outside unit cycle, vice versa. Such a chaos is the "Yang" chaos in nonlinear discrete dynamical systems.*

(ii) *The unstable chaos generated by the limit state of the unstable* $P_+^{(2^lN)}$-1 *solutions* ($l \to \infty$) *in sense of* $\mathbf{g}(\mathbf{x}_k, \mathbf{x}_{k+2^lN}, \boldsymbol{\lambda}) = \mathbf{0}$ *with all eigenvalue distribution of* $DP_+^{(2^lN)}$ *outside the unit cycle is the stable chaos generated by the limit state of the stable* $P_-^{(2^lN)}$-1 *solution* ($l \to \infty$) *in sense of* $\mathbf{g}(\mathbf{x}_k, \mathbf{x}_{k+2^lN}, \boldsymbol{\lambda}) = \mathbf{0}$, *vice versa. Such a chaos is the "Ying" chaos in nonlinear discrete dynamical systems.*

(iii) *The unstable chaos generated by the limit state of the unstable* $P_+^{(2^lN)}$-1 *solutions* ($l \to \infty$) *in sense of* $\mathbf{g}(\mathbf{x}_k, \mathbf{x}_{k+2^lN}, \boldsymbol{\lambda}) = \mathbf{0}$ *with all eigenvalue distribution of* $DP_+^{(2^lN)}$ *inside and outside the unit cycle is the unstable chaos generated by the limit state of the unstable* $P_-^{(2^lN)}$-1 *solution* ($l \to \infty$) *in sense of* $\mathbf{g}(\mathbf{x}_k, \mathbf{x}_{k+2^lN}, \boldsymbol{\lambda}) = \mathbf{0}$ *with switching all eigenvalue distribution of* $DP_+^{(2^lN)}$ *inside and outside the unit cycle, vice versa. Such a chaos is the "Ying-Yang" chaos in nonlinear discrete dynamical systems.*

Proof The proof can be referred to Luo [2]. □

6.2 Discrete Systems with Multiple Maps

Definition 6.6 Consider a set of implicit vector functions $\mathbf{f}^{(j)}: D \to D(j = 1, 2, \ldots)$ on an open set $D \subset \mathscr{R}^n$ in an n-dimensional discrete dynamical system. For $\mathbf{x}_k$, $\mathbf{x}_{k+1} \in D$, there is a discrete relation as

$$\mathbf{f}^{(j)}(\mathbf{x}_k, \mathbf{x}_{k+1}, \mathbf{p}^{(j)}) = \mathbf{0} \text{ for } j = 1, 2, \cdots \tag{6.19}$$

where the vector function is $\mathbf{f}^{(j)} = (f_1^{(j)}, f_2^{(j)}, \cdots, f_n^{(j)})^{\mathrm{T}} \in \mathscr{R}^n$ and discrete variable vector is $\mathbf{x}_k = (x_{k1}, x_{k2}, \cdots, x_{kn})^{\mathrm{T}} \in \Omega$ with a parameter vector $\mathbf{p}^{(j)} = (p_1^{(j)}, p_2^{(j)}, \cdots, p_{m_j}^{(j)})^{\mathrm{T}} \in \mathscr{R}^{m_j}$.

Definition 6.7 Consider a set of implicit vector functions $\mathbf{f}^{(j)}: D \to D(j = 1, 2, \ldots)$ on an open set $D \subset \mathscr{R}^n$ in an n-dimensional discrete dynamical system.

(i) A set for discrete relations is defined as

$$\Phi = \{\mathbf{f}^{(j)} | \mathbf{f}^{(j)}(\mathbf{x}_k, \mathbf{x}_{k+1}, \mathbf{p}^{(j)}) = \mathbf{0}, j \in \mathbb{Z}_+; k \in \mathbb{Z}\}. \tag{6.20}$$

(ii) The positive and negative discrete sets are defined as

$$\left.\begin{aligned} \Sigma_+ &= \{\mathbf{x}_{k+i} | \mathbf{x}_{k+i} \in \mathscr{R}^n, i \in \mathbb{Z}_+\} \subset D, \text{ and} \\ \Sigma_- &= \{\mathbf{x}_{k-i} | \mathbf{x}_{k-i} \in \mathscr{R}^n, i \in \mathbb{Z}_+\} \subset D, \end{aligned}\right\} \tag{6.21}$$

respectively, and the total set of the discrete states is

$$\Sigma = \Sigma_+ \cup \Sigma_-. \tag{6.22}$$

(iii) A positive mapping for $\mathbf{f}^{(j)} \in \Phi$ is defined as

$$P_j^+ : \Sigma \to \Sigma_+ \Rightarrow P_j^+ : \mathbf{x}_k \to \mathbf{x}_{k+1}, \tag{6.23}$$

and a negative mapping is defined by

$$P_j^- : \Sigma \to \Sigma_- \Rightarrow P_j^- : \mathbf{x}_k \to \mathbf{x}_{k-1}. \tag{6.24}$$

(iv) Two sets for positive and negative mappings are defined as

$$\begin{aligned} \Theta_+ &= \{P_j^+ | P_j^+ : \mathbf{x}_k \to \mathbf{x}_{k+1} \text{ with } \mathbf{f}^{(j)}(\mathbf{x}_k, \mathbf{x}_{k+1}, \mathbf{p}^{(j)}) = \mathbf{0}, \quad j \in \mathbb{Z}_+; k \in \mathbb{Z}\} \\ \Theta_- &= \{P_j^- | P_j^- : \mathbf{x}_{k+1} \to \mathbf{x}_k \text{ with } \mathbf{f}^{(j)}(\mathbf{x}_k, \mathbf{x}_{k+1}, \mathbf{p}^{(j)}) = \mathbf{0}, \quad j \in \mathbb{Z}_+; k \in \mathbb{Z}\} \end{aligned} \tag{6.25}$$

with the total mapping sets are

$$\Theta = \Theta_+ \cup \Theta_-. \tag{6.26}$$

Definition 6.8 Consider a discrete dynamical system with a set of implicit vector functions $\mathbf{f}^{(j)}: D \to D$ $(j = 1, 2, \cdots)$. For a mapping $P_j^+ \in \Theta_+$ with N-actions and $P_j^- \in \Theta_-$ with N-actions. The resultant mapping is defined as

$$P_{j^N}^+ = \underbrace{P_j^+ \circ P_j^+ \circ \cdots \circ P_j^+}_{N} \text{ and } P_{j^N}^- = \underbrace{P_j^- \circ P_j^- \circ \cdots \circ P_j^-}_{N}. \tag{6.27}$$

Definition 6.9 Consider a discrete dynamical system with a set of implicit vector functions $\mathbf{f}^{(j)}: D \to D$ $(j = 1, 2, \cdots)$. For the m-positive mappings of $P_{j_i}^+ \in \Theta_+ (i = 1, 2, \cdots, m)$ with N_{j_i}-actions ($N_{j_i} \in \{0, \mathbb{Z}_+\}$) and the corresponding m-negative mappings of $P_{j_i}^- \in \Theta_-$ $(i = 1, 2, \cdots, m)$ with N_{j_i}-actions, the resultant nonlinear mapping cluster with pure positive or negative mappings is defined as

$$\left.\begin{aligned} P^+_{(N_{jm}\cdots N_{j_2}N_{j_1})} &= \underbrace{P^+_{j_m^{N_{jm}}} \circ \cdots \circ P^+_{j_2^{N_{j2}}} \circ P^+_{j_1^{N_{j1}}}}_{m-terms}; \\ P^-_{(N_{j_1}N_{j_2}\cdots N_{jm})} &= \underbrace{P^-_{j_1^{N_{j1}}} \circ P^-_{j_2^{N_{j2}}} \circ \cdots \circ P^-_{j_m^{N_{jm}}}}_{m-terms} \end{aligned}\right\} \tag{6.28}$$

in which at least one of mappings ($P_{j_i}^+$ and $P_{j_i}^-$) with $N_{j_i} \in \mathbb{Z}_+$ possesses a nonlinear iterative relation.

Theorem 6.4 *Consider a discrete dynamical system with a set of implicit vector functions* $\mathbf{f}^{(j)}: D \to D$ $(j = 1, 2, \cdots)$. *For the m-positive mappings of* $P_{j_i}^+ \in \Theta_+$ $(i = 1, 2, \cdots, m)$ *with* N_{j_i}*-actions* ($N_{j_i} \in \{0, \mathbb{Z}_+\}$) *and the corresponding m-negative mappings of* $P_{j_i}^- \in \Theta_-$ $(i = 1, 2, \cdots, m)$ *with* N_{j_i}*-actions, the resultant nonlinear mapping with pure positive and negative mappings*

$$\mathbf{x}_{k+\Sigma_{s=1}^m N_{js}} = P^+_{(N_{jm}\cdots N_{j_2}N_{j_1})}\mathbf{x}_k \text{ and } \mathbf{x}_k = P^-_{(N_{j_1}N_{j_2}\cdots N_{jm})}\mathbf{x}_{k+\Sigma_{s=1}^m N_{js}}, \tag{6.29}$$

and $\mathbf{x}_{k+i} = P_{j_s}^+ \mathbf{x}_{k+i-1}$ *and* $\mathbf{x}_{k+i-1} = P_{j_s}^- \mathbf{x}_{k+i}$ *can be governed by*

$$\mathbf{f}(\mathbf{x}_{k+i-1}, \mathbf{x}_{k+i}, \mathbf{p}) = \mathbf{0} \text{ for } i = 1, 2, \cdots, \Sigma_{s=1}^m N_{j_t}. \tag{6.30}$$

Suppose a differentiable, vector function $\mathbf{g} \in \mathscr{R}^n$ *possesses* $\mathbf{g}(\mathbf{x}_k, \mathbf{x}_{k+\Sigma_{s=1}^m N_{js}}, \boldsymbol{\lambda})\boldsymbol{\varphi} = \mathbf{0}$. *If the solutions* $(\mathbf{x}_k^*, \cdots, \mathbf{x}^*_{k+\Sigma_{s=1}^m N_{js}})$ *of Eq. (6.29) with* $\mathbf{g}(\mathbf{x}_k, \mathbf{x}_{k+\Sigma_{s=1}^m N_{js}}, \boldsymbol{\lambda}) = \mathbf{0}$ *exist, then the following conclusions in the sense of* $\mathbf{g}(\mathbf{x}_k, \mathbf{x}_{k+\Sigma_{s=1}^m N_{js}}, \boldsymbol{\lambda}) = \mathbf{0}$ *hold.*

(i) *The stable* $P^+_{(N_{jm}\cdots N_{j_2}N_{j_1})}$-1 *solution is the unstable* $P^-_{(N_{j1}N_{j_2}\cdots N_{jm})}$-1 *solutions with all eigenvalues of* $DP^-_{(N_{j_1}N_{j_2}\cdots N_{jm})}(\mathbf{x}^*_{k+\Sigma_{s=1}^m N_{js}})$ *outside the unit cycle, vice versa.*

(ii) *The unstable* $P^+_{(N_{jm}\cdots N_{j_2}N_{j_1})}$-1 *solution with eigenvalues of* $DP^+_{(N_{jm}\cdots N_{j_2}N_{j_1})}(\mathbf{x}_k^*)$ *outside the unit cycle is the stable* $P^-_{(N_{j_1}N_{j_2}\cdots N_{jm})}$-1 *solutions, vice versa.*

(iii) *For the unstable* $P^{+}_{(N_{jm}\cdots N_{j_2}N_{j_1})}$-1 *solution with eigenvalue distribution of* $DP^{+}_{(N_{jm}\cdots N_{j_2}N_{j_1})}(\mathbf{x}^{*}_{k})$ *inside and outside the unit cycle, the corresponding* $P^{-}_{(N_{j_1}N_{j_2}\cdots N_{jm})}$-1 *solution is also unstable with switching eigenvalue distribution of* $DP^{-}_{(N_{jm}\cdots N_{j_2}N_{j_1})}(\mathbf{x}^{*}_{k+\Sigma^{m}_{s=1}N_{js}})$ *inside and outside the unit cycle, vice versa.*

(iv) *All the bifurcations of the stable and unstable* $P^{+}_{(N_{jm}\cdots N_{j_2}N_{j_1})}$-1 *solution are all the bifurcations of the unstable and stable* $P^{-}_{(N_{j_1}N_{j_2}\cdots N_{jm})}$-1 *solution, respectively.*

Proof The proof can be referred to Luo [2]. □

The chaos generated by the period-doubling of the $P^{+}_{(N_{jm}\cdots N_{j_2}N_{j_1})}$-1 and $P^{-}_{(N_{j_1}N_{j_2}\cdots N_{jm})}$-1 solutions can be described through the following theorem.

Theorem 6.5 *Consider a discrete dynamical system with a set of implicit vector functions* $\mathbf{f}^{(j)}: D \to D$ $(j = 1, 2, \cdots)$. *For the m-positive mappings of* $P^{+}_{j_i} \in \Theta_{+}$ $(i = 1, 2, \cdots, m)$ *with* N_{j_i}*-actions* $(N_{j_i} \in \{0, \mathbb{Z}_{+}\})$ *and the corresponding m-negative mappings of* $P^{-}_{j_i} \in \Theta_{-}$ $(i = 1, 2, \cdots, m)$ *with* N_{j_i}*-actions, the resultant nonlinear mapping with pure positive and negative mappings*

$$\mathbf{x}_{k+\Sigma^{m}_{s=1}N_{js}} = P^{+}_{(N_{jm}\cdots N_{j_2}N_{j_1})}\mathbf{x}_{k} \text{ and } \mathbf{x}_{k} = P^{-}_{(N_{j_1}N_{j_2}\cdots N_{jm})}\mathbf{x}_{k+\Sigma^{m}_{s=1}N_{js}}; \tag{6.31}$$

and $\mathbf{x}_{k+i} = P^{+}_{j_s}\mathbf{x}_{k+i-1}$ *and* $\mathbf{x}_{k+i-1} = P^{-}_{j_s}\mathbf{x}_{k+i}$ *can be governed by*

$$\mathbf{f}^{(j)}(\mathbf{x}_{k+i-1}, \mathbf{x}_{k+i}, \mathbf{p}^{(j)}) = \mathbf{0} \text{ for } i = 1, 2, \cdots, \Sigma^{m}_{s=1}N_{j_t}. \tag{6.32}$$

Suppose a differentiable, vector function $\mathbf{g} \in \mathscr{R}^{n}$ *possesses* $\mathbf{g}(\mathbf{x}_{k}, \mathbf{x}_{k+\Sigma^{m}_{s=1}N_{js}}, \boldsymbol{\lambda}) = \mathbf{0}$. *If the period-doubling cascade of the* $P^{+}_{(N_{jm}\cdots N_{j_2}N_{j_1})}$-1 *and* $P^{-}_{(N_{jm}\cdots N_{j_2}N_{j_1})}$-1 *solution occurs, the corresponding mapping structures are given by*

$$\left.\begin{aligned} \mathbf{x}_{k+2\Sigma^{m}_{s=1}N_{js}} &= P^{+}_{(N_{jm}\cdots N_{j_2}N_{j_1})} \circ P^{+}_{(N_{jm}\cdots N_{j_2}N_{j_1})}\mathbf{x}_{k} \\ &= P^{+}_{2(N_{jm}\cdots N_{j_2}N_{j_1})}\mathbf{x}_{k} \\ \mathbf{g}(\mathbf{x}_{k}, &\mathbf{x}_{k+2\Sigma^{m}_{s=1}N_{js}}, \boldsymbol{\lambda}) = \mathbf{0}; \end{aligned}\right\}$$
$$\left.\begin{aligned} \mathbf{x}_{k+2^{2}\Sigma^{m}_{s=1}N_{js}} &= P^{+}_{2(N_{jm}\cdots N_{j_2}N_{j_1})} \circ P^{+}_{2(N_{jm}\cdots N_{j_2}N_{j_1})}\mathbf{x}_{k} \\ &= P^{+}_{2^{2}(N_{jm}\cdots N_{j_2}N_{j_1})}\mathbf{x}_{k} \\ \mathbf{g}(\mathbf{x}_{k}, &\mathbf{x}_{k+2^{2}\Sigma^{m}_{s=1}N_{js}}, \boldsymbol{\lambda}) = \mathbf{0}; \end{aligned}\right\} \tag{6.33}$$
$$\vdots$$
$$\left.\begin{aligned} \mathbf{x}_{k+2^{l}\Sigma^{m}_{s=1}N_{js}} &= P^{+}_{2^{l-1}(N_{jm}\cdots N_{j_2}N_{j_1})} \circ P^{+}_{2^{l-1}(N_{jm}\cdots N_{j_2}N_{j_1})}\mathbf{x}_{k} \\ &= P^{+}_{2^{l}(N_{jm}\cdots N_{j_2}N_{j_1})}\mathbf{x}_{k} \\ \mathbf{g}(\mathbf{x}_{k}, &\mathbf{x}_{k+2^{l}\Sigma^{m}_{s=1}N_{js}}, \boldsymbol{\lambda}) = \mathbf{0}; \end{aligned}\right\}$$

for positive mappings and

$$\left.\begin{aligned}\mathbf{x}_k &= P^{-}_{(N_{j_1}N_{j_2}\cdots N_{jm})} \circ P^{-}_{(N_{j_1}N_{j_2}\cdots N_{jm})}\mathbf{x}_{k+2\Sigma_{s=1}^{m}N_{js}} \\ &= P^{-}_{2(N_{j_1}N_{j_2}\cdots N_{jm})}\mathbf{x}_{k+2\Sigma_{s=1}^{m}N_{js}} \\ \mathbf{g}(\mathbf{x}_k, \mathbf{x}_{k+2\Sigma_{s=1}^{m}N_{js}}, \boldsymbol{\lambda}) &= \mathbf{0};\end{aligned}\right\}$$
$$\left.\begin{aligned}\mathbf{x}_k &= P^{-}_{2(N_{j_1}N_{j_2}\cdots N_{jm})} \circ P^{-}_{2(N_{j_1}N_{j_2}\cdots N_{jm})}\mathbf{x}_{k+2^2\Sigma_{s=1}^{m}N_{js}} \\ &= P^{-}_{2^2(N_{j_1}N_{j_2}\cdots N_{jm})}\mathbf{x}_{k+2^2\Sigma_{s=1}^{m}N_{js}} \\ \mathbf{g}(\mathbf{x}_k, \mathbf{x}_{k+2^2\Sigma_{s=1}^{m}N_{js}}, \boldsymbol{\lambda}) &= \mathbf{0};\end{aligned}\right\} \tag{6.34}$$
$$\vdots$$
$$\left.\begin{aligned}\mathbf{x}_k &= P^{-}_{2^{l-1}(N_{j_1}N_{j_2}\cdots N_{jm})} \circ P^{-}_{2^{l-1}(N_{j_1}N_{j_2}\cdots N_{jm})}\mathbf{x}_{k+2^l\Sigma_{s=1}^{m}N_{js}} \\ &= P^{-}_{2^l(N_{j_1}N_{j_2}\cdots N_{jm})}\mathbf{x}_{k+2^l\Sigma_{s=1}^{m}N_{js}} \\ \mathbf{g}(\mathbf{x}_k, \mathbf{x}_{k+2^l\Sigma_{s=1}^{m}N_{js}}, \boldsymbol{\lambda}) &= \mathbf{0};\end{aligned}\right\}$$

for negative mapping, then the following statements hold, i.e.,

(i) *The stable chaos generated by the limit state of the stable* $P^{+}_{2^l(N_{jm}\cdots N_{j_2}N_{j_1})}$-1 *solutions* $(l \to \infty)$ *in sense of* $\mathbf{g}(\mathbf{x}_k, \mathbf{x}_{k+2^l\Sigma_{s=1}^{m}N_{js}}, \boldsymbol{\lambda}) = \mathbf{0}$ *is the unstable chaos generated by the limit state of the unstable stable* $P^{-}_{2^l(N_{j_1}N_{j_2}\cdots N_{jm})}$-1 *solution* $(l \to \infty)$ *in sense of* $\mathbf{g}(\mathbf{x}_k, \mathbf{x}_{k+2^l\Sigma_{s=1}^{m}N_{js}}, \boldsymbol{\lambda}) = \mathbf{0}$ *with all eigenvalue distribution of* $DP^{-}_{2^l(N_{j_1}N_{j_2}\cdots N_{jm})}$ *outside unit cycle, vice versa. Such a chaos is the "Yang" chaos in nonlinear discrete dynamical systems.*

(ii) *The unstable chaos generated by the limit state of the unstable* $P^{+}_{2^l(N_{jm}\cdots N_{j_2}N_{j_1})}$-1 *solutions* $(l \to \infty)$ *in sense of* $\mathbf{g}(\mathbf{x}_k, \mathbf{x}_{k+2^l\Sigma_{s=1}^{m}N_{js}}, \boldsymbol{\lambda}) = \mathbf{0}$ *with all eigenvalue distribution of* $P^{+}_{2^l(N_{jm}\cdots N_{j_2}N_{j_1})}$-1 *outside the unit cycle is the stable chaos generated by the limit state of the stable* $P^{-}_{2^l(N_{j_1}N_{j_2}\cdots N_{jm})}$-1 *solution* $(l \to \infty)$ *in sense of* $\mathbf{g}(\mathbf{x}_k, \mathbf{x}_{k+2^l\Sigma_{s=1}^{m}N_{js}}, \boldsymbol{\lambda}) = \mathbf{0}$*, vice versa. Such a chaos is the "Ying" chaos in nonlinear discrete dynamical systems.*

(iii) *The unstable chaos generated by the limit state of the unstable* $P^{+}_{2^l(N_{jm}\cdots N_{j_2}N_{j_1})}$-1 *solutions* $(l \to \infty)$ *in sense of* $\mathbf{g}(\mathbf{x}_k, \mathbf{x}_{k+2^l\Sigma_{s=1}^{m}N_{js}}, \boldsymbol{\lambda}) = \mathbf{0}$ *with all eigenvalue distribution of* $DP^{+}_{2^l(N_{jm}\cdots N_{j_2}N_{j_1})}$ *inside and outside the unit cycle is the unstable chaos generated by the limit state of the unstable* $P^{-}_{2^l(N_{j_1}N_{j_2}\cdots N_{jm})}$-1 *solution* $(l \to \infty)$ *in sense of* $\mathbf{g}(\mathbf{x}_k, \mathbf{x}_{k+2^l\Sigma_{s=1}^{m}N_{js}}, \boldsymbol{\lambda}) = \mathbf{0}$ *with switching all eigenvalue distribution of* $DP^{-}_{2^l(N_{j_1}N_{j_2}\cdots N_{jm})}$ *inside and outside the unit cycle, vice versa. Such a chaos is the "Ying-Yang" chaos in nonlinear discrete dynamical systems.*

Proof The proof can be referred to Luo [2]. □

6.3 Complete Dynamics of a Henon Map System

As in Luo and Guo [3], consider the Henon map system as

$$\left.\begin{aligned} f_1(\mathbf{x}_k, \mathbf{x}_{k+1}, \mathbf{p}) &= x_{k+1} - y_k - 1 + a x_k^2 = 0, \\ f_2(\mathbf{x}_k, \mathbf{x}_{k+1}, \mathbf{p}) &= y_{k+1} - b x_k = 0, \end{aligned}\right\} \tag{6.35}$$

where $\mathbf{x}_k = (x_k, y_k)^{\mathrm{T}}$, $\mathbf{f} = (f_1, f_2)^{\mathrm{T}}$ and $\mathbf{p} = (a, b)^{\mathrm{T}}$. Consider two positive and negative mapping structures as

$$\begin{aligned} \mathbf{x}_{k+N} &= P_+^{(N)} \mathbf{x}_k = \underbrace{P_+ \circ \cdots \circ P_+ \circ P_+}_{N-\text{terms}} \mathbf{x}_k, \\ \mathbf{x}_k &= P_-^{(N)} \mathbf{x}_{k+N} = \underbrace{P_- \circ \cdots \circ P_- \circ P_-}_{N-\text{terms}} \mathbf{x}_{k+N}. \end{aligned} \tag{6.36}$$

Equations (6.35) and (6.36) give

$$\left.\begin{aligned} &\mathbf{f}(\mathbf{x}_k, \mathbf{x}_{k+1}, \mathbf{p}) = \mathbf{0}, \\ &\mathbf{f}(\mathbf{x}_{k+1}, \mathbf{x}_{k+2}, \mathbf{p}) = \mathbf{0}, \\ &\vdots \\ &\mathbf{f}(\mathbf{x}_{k+N-1}, \mathbf{x}_{k+N}, \mathbf{p}) = \mathbf{0} \end{aligned}\right\} \tag{6.37}$$

and

$$\left.\begin{aligned} &\mathbf{f}(\mathbf{x}_{k+N-1}, \mathbf{x}_{k+N}, \mathbf{p}) = \mathbf{0}, \\ &\mathbf{f}(\mathbf{x}_{k+N-2}, \mathbf{x}_{k+N-1}, \mathbf{p}) = \mathbf{0}, \\ &\vdots \\ &\mathbf{f}(\mathbf{x}_k, \mathbf{x}_{k+1}, \mathbf{p}) = \mathbf{0}. \end{aligned}\right\} \tag{6.38}$$

The switching of equation order in Eq. (6.38) shows Eqs. (6.37) and (6.38) are identical. For periodic solutions of the positive and negative maps, the periodicity of the positive and negative mapping structures of the Henon map requires

$$\mathbf{x}_{k+N} = \mathbf{x}_k \text{ or } \mathbf{x}_k = \mathbf{x}_{k+N}. \tag{6.39}$$

So the periodic solutions $\mathbf{x}_{k+j}^*$ $(j = 0, 1, \cdots, N)$ for the negative and positive mapping structures are the same, which are given by solving Eqs. (6.37) and (6.38) with Eq. (6.39). However, the stability and bifurcation are different because $\mathbf{x}_{k+j}$ varies with $\mathbf{x}_{k+j-1}$ for the jth positive mapping and $\mathbf{x}_{k+j-1}$ varies with $\mathbf{x}_{k+j}$ for the

jth negative mapping. For a small perturbation, Eq. (6.37) for the positive mapping gives

$$[\frac{\partial \mathbf{f}}{\partial \mathbf{x}_{k+j-1}}] + [\frac{\partial \mathbf{f}}{\partial \mathbf{x}_{k+j}}] \cdot [\frac{\partial \mathbf{x}_{k+j}}{\partial \mathbf{x}_{k+j-1}}]|_{(\mathbf{x}^*_{k+j-1}, \mathbf{x}^*_{k+j})} = \mathbf{0}, \tag{6.40}$$

where

$$\begin{aligned}[\frac{\partial \mathbf{f}}{\partial \mathbf{x}_{k+j-1}}]_{(\mathbf{x}^*_{k+j-1}, \mathbf{x}^*_{k+j})} &= \begin{bmatrix} \dfrac{\partial f_1}{\partial x_{k+j-1}} & \dfrac{\partial f_1}{\partial y_{k+j-1}} \\ \dfrac{\partial f_2}{\partial x_{k+j-1}} & \dfrac{\partial f_2}{\partial y_{k+j-1}} \end{bmatrix}_{(\mathbf{x}^*_{k+j-1}, \mathbf{x}^*_{k+j})} \\ &= \begin{bmatrix} 2ax^*_{k+j-1} & -1 \\ -b & 0 \end{bmatrix},\end{aligned} \tag{6.41}$$

$$\begin{aligned}[\frac{\partial \mathbf{f}}{\partial \mathbf{x}_{k+j}}]_{(\mathbf{x}^*_{k+j-1}, \mathbf{x}^*_{k+j})} &= \begin{bmatrix} \dfrac{\partial f_1}{\partial x_{k+j}} & \dfrac{\partial f_1}{\partial y_{k+j}} \\ \dfrac{\partial f_2}{\partial x_{k+j}} & \dfrac{\partial f_2}{\partial y_{k+j}} \end{bmatrix}_{(\mathbf{x}^*_{k+j-1}, \mathbf{x}^*_{k+j})} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.\end{aligned} \tag{6.42}$$

So

$$\begin{aligned} DP_+(\mathbf{x}^*_{k+j-1}) = [\frac{\partial \mathbf{x}_{k+j}}{\partial \mathbf{x}_{k+j-1}}]_{\mathbf{x}^*_{k+j-1}} &= [\frac{\partial \mathbf{f}}{\partial \mathbf{x}_{k+j}}]^{-1} [\frac{\partial \mathbf{f}}{\partial \mathbf{x}_{k+j-1}}]_{\mathbf{x}^*_{k+j-1}} \\ &= \begin{bmatrix} 2ax^*_{k+j-1} & -1 \\ -b & 0 \end{bmatrix}. \end{aligned} \tag{6.43}$$

Similarly, for the negative mapping,

$$[\frac{\partial \mathbf{f}}{\partial \mathbf{x}_{k+j}}] + [\frac{\partial \mathbf{f}}{\partial \mathbf{x}_{k+j-1}}] \cdot [\frac{\partial \mathbf{x}_{k+j-1}}{\partial \mathbf{x}_{k+j}}]|_{(\mathbf{x}^*_{k+j-1}, \mathbf{x}^*_{k+j})} = \mathbf{0}. \tag{6.44}$$

With Eqs. (6.41) and (6.42), the foregoing equation gives

$$\begin{aligned} DP_-(\mathbf{x}^*_{k+j}) = [\frac{\partial \mathbf{x}_{k+j-1}}{\partial \mathbf{x}_{k+j}}]_{\mathbf{x}^*_{k+j}} &= [\frac{\partial \mathbf{f}}{\partial \mathbf{x}_{k+j-1}}]^{-1} [\frac{\partial \mathbf{f}}{\partial \mathbf{x}_{k+j}}]_{\mathbf{x}^*_{k+j}} \\ &= -\frac{1}{b} \begin{bmatrix} 0 & 1 \\ b & 2ax^*_{k+j-1} \end{bmatrix}. \end{aligned} \tag{6.45}$$

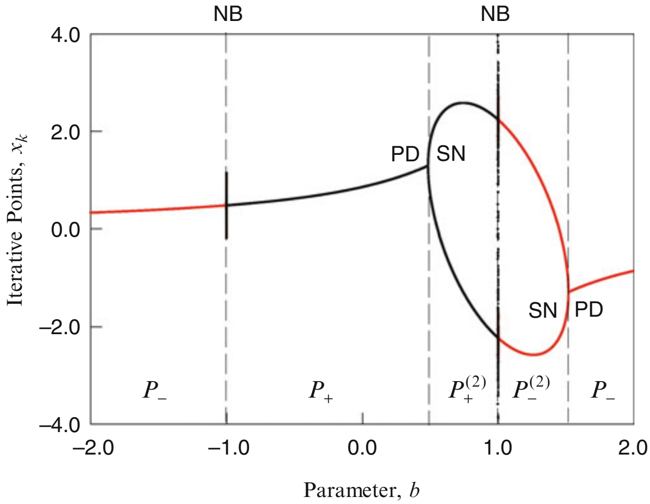

Fig. 6.2 Numerical predictions of periodic solutions of the Henon mapping with negative and positive mappings ($a = 0.2$)

Thus, the resultant perturbation of the mapping structure in Eq. (6.36) gives

$$\begin{aligned} \delta\mathbf{x}_{k+N} &= DP_+^{(N)}\delta\mathbf{x}_k = \underbrace{DP_+\cdot\ldots\cdot DP_+\cdot DP_+}_{N-\text{terms}}\delta\mathbf{x}_k, \\ \delta\mathbf{x}_k &= DP_-^{(N)}\delta\mathbf{x}_{k+N} = \underbrace{DP_-\cdot\ldots\cdot DP_-\cdot DP_-}_{N-\text{terms}}\delta\mathbf{x}_{k+N}, \end{aligned} \tag{6.46}$$

where

$$\left.\begin{aligned} DP_+^{(N)} &= \prod\nolimits_{j=1}^{N} DP_+(\mathbf{x}^*_{k+N-j}), \\ DP_-^{(N)} &= \prod\nolimits_{j=1}^{N} DP_-(\mathbf{x}^*_{k+N-j+1}). \end{aligned}\right\} \tag{6.47}$$

From the resultant Jacobian matrix, the eigenvalue analysis can be completed. Before analytical prediction of periodic solution, a numerical prediction of the periodic solutions of the Henon map is presented with varying parameter b for $a = 0.2$, as shown in Fig. 6.2. The dashed vertical lines give the bifurcation points. The acronyms “PD,” “SN,” and “NB” represented the period-doubling bifurcation, saddle-stable node bifurcation, and Neimark bifurcation, respectively.

From the numerical prediction, the stable periodic solutions of the Henon map are obtained. Herein, through the corresponding mapping structures, the stable and unstable periodic solutions for positive and negative mappings of the Henon maps are presented in Fig. 6.3. The acronyms “PD”, “SN”, and “NB” represented the period-doubling bifurcation, saddle-stable node bifurcation, and Neimark

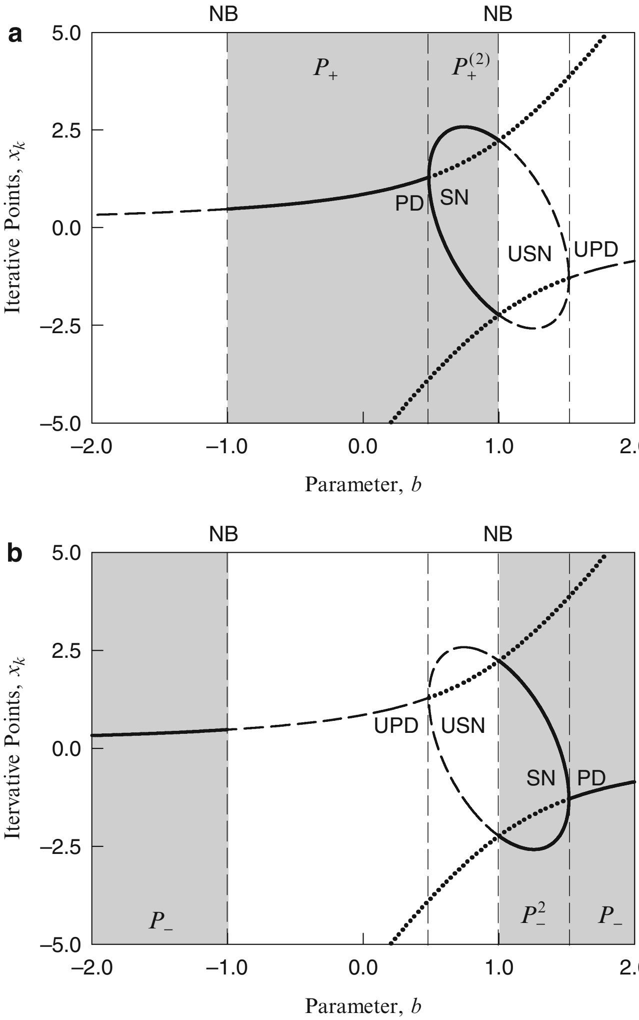

Fig. 6.3 Analytical predictions of *stable* and *unstable* periodic solutions of the Henon map: (**a**) positive mapping (P_+) and (**b**) positive mapping (P_-) ($a = 0.2$ and $b \in (-\infty, +\infty)$)

bifurcation, respectively. The acronyms "UPD" and "USN" represented the period-doubling bifurcation relative to unstable nodes and saddle-unstable node bifurcation, respectively. From the eigenvalue analysis, the stable periodic solutions for positive mapping P_+ lie in $b \in (-1.0, 1.0)$, which is the same as the numerical prediction. In other words, the stable period-1 solution of P_+ is in $b \in (-1, 0.4805)$. For $b \in (0.4805, +\infty)$, the unstable period-1 solution of P_+ is saddle. For $b \in (-\infty, -1.0)$, the unstable period-1 solution of P_+ is unstable focus. The corresponding bifurcations are Neimark bifurcation (NB) and period-doubling bifurcation (PD). However, another unstable period-1 solution of P_+ exists.

For $b \in (1.5215, +\infty)$, the unstable periodic solution of P_+ is unstable node. However, for $b \in (-\infty, 1.5215)$, the unstable periodic solution of P_+ is saddle. Thus, the unstable period-doubling bifurcation (UPD) of the period-1 solution of P_+ occurs at $b \approx 1.5215$. At this point, the unstable periodic solution is from an unstable node to saddle. Because of the unstable period-doubling bifurcation, the unstable periodic solution of $P_+^{(2)}$ is obtained for $b \in (1.0, 1.5215)$. This unstable periodic solution is from unstable focus to unstable node during the parameter of $b \in (1.0, 1.5215)$. At $b \approx 1.5215$, the bifurcation of the unstable periodic solution of $P_+^{(2)}$ occurs between the saddle and unstable node. This bifurcation is called the unstable saddle-node bifurcation. At $b = 1.0$, the Neimark bifurcation (NB) between the periodic solutions of $P_+^{(2)}$ pertaining to the unstable and stable focuses occurs. The stable periodic solution of $P_+^{(2)}$ is from the stable node to the stable focus for $b \in (0.4805, 1.0)$.

Again, from the eigenvalue analysis, the stable periodic solutions for positive mapping P_- lie in $b \in (-\infty, -1.0)$ and $b \in (1.0, +\infty)$, which is the same as in numerical prediction. The stable period-1 solution of P_- is stable focuses in $b \in (-\infty, -1.0)$ and stable nodes in $b \in (1.5215, +\infty)$. For $b \in (-1.0, 0.4805)$, the unstable period-1 solution of P_- is from the unstable focus to unstable node. At $b = -1$, the bifurcation between the stable and unstable period-1 solution of P_- is the Neimark bifurcation (NB). For $b \in (0.4805, +\infty)$, the unstable period-1 solution of P_- is saddle. Thus, the bifurcation between the period-1 solution of P_- between the unstable node and saddle occurs at $b = 0.4805$, which is called the unstable period-doubling bifurcation (UPD). For $b \in (0.4805, +1)$, the unstable period-2 solution of P_- (i.e., $P_-^{(2)}$) is from the unstable node to the unstable focus. For $b \in (1.0, 1.5215)$, the stable period-2 solution of P_- (i.e., $P_-^{(2)}$) is from the stable focus to the stable nodes. Thus, the point at $b \approx 0.4805$ is the bifurcation of the unstable periodic solution of $P_-^{(2)}$ which is the unstable saddle-node bifurcation between the unstable node and saddle (i.e., USN). For the point at $b = 1$, the Neimark bifurcation between the periodic solutions of $P_-^{(2)}$ relative to the unstable and stable focuses occurs. The point at $b \approx 1.5215$ is the bifurcation of the stable periodic solution of $P_-^{(2)}$ which is the saddle bifurcation between the stable node and saddle (SN). For $b \in (-\infty, 1.5215)$, the unstable period-1 solution of P_- is saddle. At $b \approx 1.5215$, the period-doubling bifurcation (PD) of the period-1 solution of P_- takes place.

The strange attractors caused by the period-doubling bifurcation cascade were presented by many researchers. Herein, the strange attractors relative to the Neimark bifurcation between the periodic solutions relative to the unstable and stable focuses are presented. The Poincare mapping relative to the Neimark bifurcation of the period-1 and period-2 solutions of positive mapping (or negative mapping) at $a = 0.2$ and $b = \pm 1$ is presented in Fig. 6.4. In Fig. 6.4a, the most inside point $(x_k^*, y_k^*) \approx (0.4772, -0.4772)$ is the point for the period-1 solution of P_+ or P_- relative to the Neimark bifurcation. With the initial condition $(x_k^*, y_k^*) \approx (1.7188, 0.0)$, the most outside curve is the biggest boundary for the strange attractors around the period-1 solutions with the Neimark bifurcation. The skew symmetry of the strange attractors in the Poincare mapping section is observed. In Fig. 6.4b, the two points $(x_k^*, y_k^*) \approx (2.2361, -2.2361)$

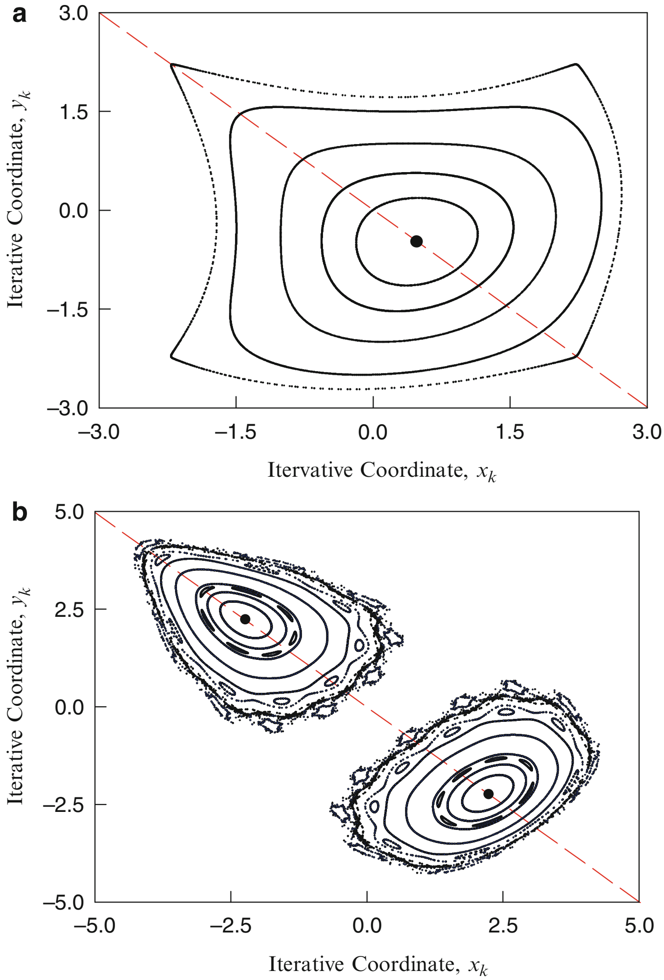

Fig. 6.4 Poincare mappings of the Henon map for the Neimark bifurcation: (**a**) period-1 solution (i.e., P_+-1 or P_--1) ($a = 0.2$ and $b = -1$), and (**b**) period-2 solution (i.e., $P_+^{(2)}$-1 or $P_-^{(2)}$-1) ($a = 0.2$ and $b = 1$)

and $(-2.2361, 2.2361)$ are the points for the period-2 solution of P_+ or P_- relative to the Neimark bifurcation. With the outer chaotic layer, the strange attractor near the periodic solutions of $P_+^{(2)}$-1 (or $P_+^{(2)}$-1) disappears. This chaotic layer possesses eight islands inside the barrier and nine islands outside the barrier. For $(x_k, y_k) \approx (2.9397, -2.2361)$, the seven islands are observed. The skew symmetry of the strange attractors in the Poincare mapping section is observed.

6.4 Companion and Synchronization

This section will extend the concepts presented in the previous section. The companion and synchronization of two discrete dynamical systems will be presented.

Definition 6.10 Consider the αth implicit vector function $\mathbf{f}^{(\alpha)}: D \to D$ $(\alpha = 1, 2, \ldots, N)$ on an open set $D \subset \mathscr{R}^n$ in an n-dimensional discrete dynamical system. For $\mathbf{x}_k, \mathbf{x}_{k+1} \in D$, there is a discrete relation as

$$\mathbf{f}^{(\alpha)}(\mathbf{x}_k, \mathbf{x}_{k+1}, \mathbf{p}^{(\alpha)}) = \mathbf{0}, \tag{6.48}$$

where the vector function is $\mathbf{f}^{(\alpha)} = (f_1^{(\alpha)}, f_2^{(\alpha)}, \cdots, f_n^{(\alpha)})^{\mathrm{T}} \in \mathscr{R}^n$ and discrete variable vector is $\mathbf{x}_k = (x_{k1}, x_{k2}, \cdots, x_{kn})^{\mathrm{T}} \in D$ with the corresponding parameter vector $\mathbf{p}^{(\alpha)} = (p_1^{(\alpha)}, p_2^{(\alpha)}, \cdots, p_{m_\alpha}^{(\alpha)})^{\mathrm{T}} \in \mathscr{R}^{m_\alpha}$.

Similarly, the discrete sets, positive and negative mappings for discrete dynamical system of $\mathbf{f}^{(\alpha)}(\mathbf{x}_k, \mathbf{x}_{k+1}, \mathbf{p}^{(\alpha)}) = \mathbf{0}$ in Eq. (6.48) are defined.

Definition 6.11 For a discrete dynamical system in Eq. (6.48), the positive and negative discrete sets are defined by

$$\left.\begin{aligned} \Sigma_+^{(\alpha)} &= \{\mathbf{x}_{k+i}^{(\alpha)} | \mathbf{x}_{k+i}^{(\alpha)} \in \mathscr{R}^n, i \in \mathbb{Z}_+\} \subset D \text{ and} \\ \Sigma_-^{(\alpha)} &= \{\mathbf{x}_{k-i}^{(\alpha)} | \mathbf{x}_{k-i}^{(\alpha)} \in \mathscr{R}^n, i \in \mathbb{Z}_+\} \subset D, \end{aligned}\right\} \tag{6.49}$$

respectively. The corresponding discrete set is

$$\Sigma^{(\alpha)} = \Sigma_+^{(\alpha)} \cup \Sigma_-^{(\alpha)}. \tag{6.50}$$

A positive mapping for discrete dynamical system is defined as

$$P_{\alpha+}: \Sigma^{(\alpha)} \to \Sigma_+^{(\alpha)} \Rightarrow P_{\alpha+}: \mathbf{x}_k^{(\alpha)} \to \mathbf{x}_{k+1}^{(\alpha)}, \tag{6.51}$$

and a negative mapping is defined by

$$P_{\alpha-}: \Sigma^{(\alpha)} \to \Sigma_-^{(\alpha)} \Rightarrow P_{\alpha-}: \mathbf{x}_k^{(\alpha)} \to \mathbf{x}_{k-1}^{(\alpha)}. \tag{6.52}$$

Definition 6.12 For two discrete dynamical systems in Eq. (6.48), consider two points $\mathbf{x}_k^{(\alpha)}, \mathbf{x}_k^{(\beta)} \in D$ and $\mathbf{x}_{k+1}^{(\alpha)}, \mathbf{x}_{k+1}^{(\beta)} \in D$, and there is a specific, differentiable, vector function $\boldsymbol{\varphi} = (\varphi_1, \varphi_2, \cdots, \varphi_l)^{\mathrm{T}} \in \mathscr{R}^l$. For a small number $\varepsilon_k > 0$, there is a small number $\varepsilon_{k+1} > 0$. Suppose there are two sub-domains $\mathrm{U}_k^{(\alpha)} \subset D$ and $\mathrm{U}_k^{(\beta)} \subset D$, then for $\mathbf{x}_k^{(\alpha)} \in \mathrm{U}_k^{(\alpha)}$ and $\mathbf{x}_k^{(\beta)} \in \mathrm{U}_k^{(\beta)}$,

$$||\boldsymbol{\varphi}(\mathbf{x}_k^{(\alpha)}, \mathbf{x}_k^{(\beta)}, \boldsymbol{\lambda})|| \leq \varepsilon_k. \tag{6.53}$$

(i) For $\varepsilon_{k+1}>0$, there are two sub-domains $\mathrm{U}_{k+1}^{(\alpha)} \subset D$ and $\mathrm{U}_{k+1}^{(\beta)} \subset D$. If for $\mathbf{x}_{k+1}^{(\alpha)} \in \mathrm{U}_{k+1}^{(\alpha)}$ and $\mathbf{x}_{k+1}^{(\beta)} \in \mathrm{U}_{k+1}^{(\beta)}$

$$||\boldsymbol{\varphi}(\mathbf{x}_{k+1}^{(\alpha)}, \mathbf{x}_{k+1}^{(\beta)}, \boldsymbol{\lambda})|| \leq \varepsilon_{k+1}, \tag{6.54}$$

then, the discrete dynamical systems of $\mathbf{f}^{(\alpha)}$ and $\mathbf{f}^{(\beta)}$ are called the companion in sense of $\boldsymbol{\varphi}$ during the kth and $(k+1)$th iteration.

(i.a) The discrete dynamical systems of $\mathbf{f}^{(\alpha)}$ and $\mathbf{f}^{(\beta)}$ is called the finite companion if for $\mathbf{x}_{k+j}^{(\alpha)} \in \mathrm{U}_{k+j}^{(\alpha)} \subset D$ and $\mathbf{x}_{k+j}^{(\beta)} \in \mathrm{U}_{k+j}^{(\beta)} \subset D$

$$||\boldsymbol{\varphi}(\mathbf{x}_{k+j}^{(\alpha)}, \mathbf{x}_{k+j}^{(\beta)}, \boldsymbol{\lambda})|| \leq \varepsilon_{k+j} \quad \text{for } j = 1, 2, \cdots, N. \tag{6.55}$$

(i.b) The discrete dynamical systems of $\mathbf{f}^{(\alpha)}$ and $\mathbf{f}^{(\beta)}$ is called the absolute permanent companion if $\mathbf{x}_{k+j}^{(\alpha)} \in \mathrm{U}_{k+j}^{(\alpha)} \subset D$ and $\mathbf{x}_{k+j}^{(\beta)} \in \mathrm{U}_{k+j}^{(\beta)} \subset D$

$$||\boldsymbol{\varphi}(\mathbf{x}_{k+j}^{(\alpha)}, \mathbf{x}_{k+j}^{(\beta)}, \boldsymbol{\lambda})|| \leq \varepsilon_{k+j} \quad \text{for } j = 1, 2, \cdots. \tag{6.56}$$

(i.c) The discrete dynamical systems of $\mathbf{f}^{(\alpha)}$ and $\mathbf{f}^{(\beta)}$ is called the repeatable finite companion if $\mathbf{x}_{k+jN(-)}^{(\alpha)} \in \mathrm{U}_{k+jN(-)}^{(\alpha)} \subset D$ and $\mathbf{x}_{k+j(-)}^{(\beta)} \in \mathrm{U}_{k+j(-)}^{(\beta)} \subset D$

$$\begin{aligned}
&\Delta\mathbf{I}^{(\alpha)} : \mathbf{x}_{k+jN(-)}^{(\alpha)} \to \mathbf{x}_{k+jN(+)}^{(\alpha)}, \text{ and } \Delta\mathbf{I}^{(\beta)} : \mathbf{x}_{k+jN(-)}^{(\beta)} \to \mathbf{x}_{k+jN(+)}^{(\beta)},\\
&\mathbf{x}_{k+jN(+)}^{(\alpha)} = \mathbf{x}_{k+jN(-)}^{(\alpha)} + \Delta\mathbf{I}_{jN}^{(\alpha)} \text{ and } \mathbf{x}_{k+jN(+)}^{(\beta)} = \mathbf{x}_{k+jN(-)}^{(\beta)} + \Delta\mathbf{I}_{jN}^{(\beta)};\\
&||\boldsymbol{\varphi}(\mathbf{x}_{k+j(+)}^{(\alpha)}, \mathbf{x}_{k+j(+)}^{(\beta)}, \boldsymbol{\lambda})|| \leq \varepsilon_{k+\mathrm{mod}(j,N)} \text{ for } j = 1, 2, \cdots,\\
&\text{with } \mathbf{x}_{k+j(+)}^{(\alpha)} \in \mathrm{U}_{k+\mathrm{mod}(j,N)}^{(\alpha)} \text{ and } \mathbf{x}_{k+j(+)}^{(\beta)} \in \mathrm{U}_{k+\mathrm{mod}(j,N)}^{(\beta)}.
\end{aligned} \tag{6.57}$$

(ii) For $\varepsilon_k>0$, $\varepsilon_{k+(N_\alpha:N_\beta)}>0$ there are there are two sub-domains $\mathrm{U}_{k+N_\alpha}^{(\alpha)} \subset D$ and $\mathrm{U}_{k+N_\beta}^{(\beta)} \subset D$. For $\mathbf{x}_{k+N_\alpha}^{(\alpha)} \in \mathrm{U}_{k+N_\alpha}^{(\alpha)}$ and $\mathbf{x}_{k+N_\beta}^{(\beta)} \in \mathrm{U}_{k+N_\beta}^{(\beta)}$ if

$$||\boldsymbol{\varphi}(\mathbf{x}_{k+N_\alpha}^{(\alpha)}, \mathbf{x}_{k+N_\beta}^{(\beta)}, \boldsymbol{\lambda})|| \leq \varepsilon_{k+(N_\alpha:N_\beta)}, \tag{6.58}$$

then the discrete dynamical systems of $\mathbf{f}^{(\alpha)}$ from the kth to $(k+N_\alpha)$th iteration and $\mathbf{f}^{(\beta)}$ from the kth to $(k+N_\beta)$th iteration are called the $(N_\alpha : N_\beta)$-companion in sense of $\boldsymbol{\varphi}$.

(ii.a) The discrete dynamical systems of $\mathbf{f}^{(\alpha)}$ and $\mathbf{f}^{(\beta)}$ is called the finite $(N_\alpha : N_\beta)$ companion if for $\mathbf{x}_{k+jN_\alpha}^{(\alpha)} \in \mathrm{U}_{k+jN_\alpha}^{(\alpha)} \subset D$ and $\mathbf{x}_{k+jN_\beta}^{(\beta)} \in \mathrm{U}_{k+jN_\beta}^{(\beta)} \subset D$

$$||\boldsymbol{\varphi}(\mathbf{x}^{(\alpha)}_{k+jN_\alpha}, \mathbf{x}^{(\beta)}_{k+jN_\beta}, \boldsymbol{\lambda})|| \le \varepsilon_{k+j(N_\alpha:N_\beta)} \text{ for } j = 1, 2, \cdots, N \tag{6.59}$$

(ii.b) The discrete dynamical systems of $\mathbf{f}^{(\alpha)}$ and $\mathbf{f}^{(\beta)}$ is called the absolute permanent $(N_\alpha : N_\beta)$ companion if $\mathbf{x}^{(\alpha)}_{k+jN_\alpha} \in \mathrm{U}^{(\alpha)}_{k+jN_\alpha} \subset D$ and $\mathbf{x}^{(\beta)}_{k+jN_\beta} \in \mathrm{U}^{(\beta)}_{k+jN_\beta} \subset D$

$$||\boldsymbol{\varphi}(\mathbf{x}^{(\alpha)}_{k+jN_\alpha}, \mathbf{x}^{(\beta)}_{k+jN_\beta}, \boldsymbol{\lambda})|| \le \varepsilon_{k+j(N_\alpha:N_\beta)} \text{ for } j = 1, 2, \cdots, \tag{6.60}$$

(ii.c) The discrete dynamical systems of $\mathbf{f}^{(\alpha)}$ and $\mathbf{f}^{(\beta)}$ is called the repeatable finite $(N_\alpha : N_\beta)$ companion if $\mathbf{x}^{(\alpha)}_{k+jN_\alpha} \in \mathrm{U}^{(\alpha)}_{k+jN_\alpha} \subset D$ and $\mathbf{x}^{(\beta)}_{k+jN_\beta} \in \mathrm{U}^{(\beta)}_{k+jN_\beta} \subset D$

$$\begin{aligned}
&\Delta\mathbf{I}^{(\alpha)} : \mathbf{x}^{(\alpha)}_{k+jN_\alpha(-)} \to \mathbf{x}^{(\alpha)}_{k+jN_\alpha(+)}, \text{ and } \Delta\mathbf{I}^{(\beta)} : \mathbf{x}^{(\beta)}_{k+jN_\beta(-)} \to \mathbf{x}^{(\beta)}_{k+jN_\beta(+)}\\
&\mathbf{x}^{(\alpha)}_{k+jN_\alpha(+)} = \mathbf{x}^{(\alpha)}_{k+jN_\alpha(-)} + \Delta\mathbf{I}^{(\alpha)}_{jN_\alpha} \text{ and } \mathbf{x}^{(\beta)}_{k+jN_\beta(+)} = \mathbf{x}^{(\beta)}_{k+jN_\beta(-)} + \Delta\mathbf{I}^{(\beta)}_{jN_\beta}\\
&||\boldsymbol{\varphi}(\mathbf{x}^{(\alpha)}_{k+jN_\alpha(+)}, \mathbf{x}^{(\beta)}_{k+jN_\beta(+)}, \boldsymbol{\lambda})|| \le \varepsilon_{k+\mathrm{mod}(j,N)(N_\alpha:N_\beta)} \text{ for } j = 1, 2, \cdots,\\
&\mathbf{x}^{(\alpha)}_{k+jN_\alpha(+)} \in \mathrm{U}^{(\alpha)}_{k+\mathrm{mod}(j,N)N_\alpha} \text{ and } \mathbf{x}^{(\beta)}_{k+jN_\beta(+)} \in \mathrm{U}^{(\beta)}_{k+\mathrm{mod}(j,N)N_\beta}.
\end{aligned} \tag{6.61}$$

Definition 6.13 For two discrete dynamical systems in Eq. (6.48), consider two points $\mathbf{x}^{(\alpha)}_k, \mathbf{x}^{(\beta)}_k \in D$ and $\mathbf{x}^{(\alpha)}_{k+1}, \mathbf{x}^{(\beta)}_{k+1} \in D$, and there is a specific, differentiable, vector function $\boldsymbol{\varphi} = (\varphi_1, \varphi_2, \cdots, \varphi_l)^{\mathrm{T}} \in \mathscr{R}^l$. For

$$\boldsymbol{\varphi}(\mathbf{x}^{(\alpha)}_k, \mathbf{x}^{(\beta)}_k, \boldsymbol{\lambda}) = \mathbf{0}, \tag{6.62}$$

(i) If

$$\boldsymbol{\varphi}(\mathbf{x}^{(\alpha)}_{k+1}, \mathbf{x}^{(\beta)}_{k+1}, \boldsymbol{\lambda}) = \mathbf{0}, \tag{6.63}$$

then, discrete dynamical systems of $\mathbf{f}^{(\alpha)}$ and $\mathbf{f}^{(\beta)}$ are called the $(1:1)$ synchronization in sense of $\boldsymbol{\varphi}$;

(ii) If

$$\begin{aligned}
&\boldsymbol{\varphi}(\mathbf{x}^{(\alpha)}_{k+1}, \mathbf{x}^{(\beta)}_{k+1}, \boldsymbol{\lambda}) = \mathbf{0} \text{ with}\\
&\Delta\mathbf{I}^{(\alpha)} : \mathbf{x}^{(\alpha)}_{k+1(-)} \to \mathbf{x}^{(\alpha)}_{k+1(+)} \text{ and } \Delta\mathbf{I}^{(\beta)} : \mathbf{x}^{(\beta)}_{k+1(-)} \to \mathbf{x}^{(\beta)}_{k+1(+)},\\
&\mathbf{x}^{(\alpha)}_{k+1(+)} = \mathbf{x}^{(\alpha)}_{k+1(-)} + \Delta\mathbf{I}^{(\alpha)} \text{ and } \mathbf{x}^{(\beta)}_{k+1(+)} = \mathbf{x}^{(\beta)}_{k+1(-)} + \Delta\mathbf{I}^{(\beta)},\\
&\mathbf{x}^{(\alpha)}_{k+1(+)} = \mathbf{x}^{(\alpha)}_k \text{ and } \mathbf{x}^{(\beta)}_{k+1(+)} = \mathbf{x}^{(\beta)}_k;
\end{aligned} \tag{6.64}$$

then, discrete dynamical systems of $\mathbf{f}^{(\alpha)}$ and $\mathbf{f}^{(\beta)}$ are called the repeatable $(1:1)$ synchronization in sense of $\boldsymbol{\varphi}$;

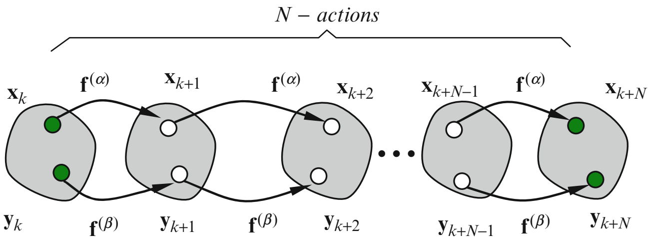

Fig. 6.5 Companion of two discrete dynamical systems

(iii) If

$$\boldsymbol{\varphi}(\mathbf{x}_{k+N_\alpha}^{(\alpha)}, \mathbf{x}_{k+N_\beta}^{(\beta)}, \boldsymbol{\lambda}) = \mathbf{0} \tag{6.65}$$

then the discrete dynamical systems of $\mathbf{f}^{(\alpha)}$ and $\mathbf{f}^{(\beta)}$ are called the $(N_\alpha : N_\beta)$-synchronization in sense of $\boldsymbol{\varphi}$;

(iv) If

$$\begin{aligned}
&\boldsymbol{\varphi}(\mathbf{x}_{k+N_\alpha}^{(\alpha)}, \mathbf{x}_{k+N_\beta}^{(\beta)}, \boldsymbol{\lambda}) = \mathbf{0} \text{ with} \\
&\Delta\mathbf{I}^{(\alpha)} : \mathbf{x}_{k+N_\alpha}^{(\alpha)} \to \mathbf{x}_k^{(\alpha)} \text{ and } \Delta\mathbf{I}^{(\beta)} : \mathbf{x}_{k+N_\beta}^{(\beta)} \to \mathbf{x}_k^{(\beta)}, \\
&\mathbf{x}_{k+N_\alpha(+)}^{(\alpha)} = \mathbf{x}_{k+N_\alpha(-)}^{(\alpha)} + \Delta\mathbf{I}^{(\alpha)} \text{ and } \mathbf{x}_{k+N_\beta(+)}^{(\beta)} = \mathbf{x}_{k+N_\beta(-)}^{(\beta)} + \Delta\mathbf{I}^{(\beta)}, \\
&\mathbf{x}_{k+N_\beta(+)}^{(\alpha)} = \mathbf{x}_k^{(\alpha)} \text{ and } \mathbf{x}_{k+N_\beta(+)}^{(\beta)} = \mathbf{x}_k^{(\beta)}.
\end{aligned} \tag{6.66}$$

then the discrete dynamical systems of $\mathbf{f}^{(\alpha)}$ and $\mathbf{f}^{(\beta)}$ are called the repeatable $(N_\alpha : N_\beta)$-synchronization in sense of $\boldsymbol{\varphi}$.

From the definition, the companions of two discrete dynamical systems are presented in Figs. 6.5 and 6.6. For each step, if the corresponding relation satisfies Eq. (6.62), the companion is called the $(1:1)$ companion, which is presented in Fig. 6.5. The shaded areas are the companion domain which is controlled by ε_k and $\boldsymbol{\varphi}$. For the repeated companion, for each step, the companion with specific impulses will have the same control domains. Such shaded areas can be overlapped or separated. The $(N_\alpha : N_\beta)$ state for $\mathbf{f}^{(\alpha)}$ with N_α-iterations and $\mathbf{f}^{(\beta)}$ with N_β-iterations satisfy Eq. (6.65) is called the $(N_\alpha : N_\beta)$-companion, which is sketched in Fig. 6.6a. This companion does not require each iteration step to do so. The companion states are shaded. For the repeated companion, the companion state with specific impulses will have the same control domains. The companion for negative maps can be similarly defined, as shown in Fig. 6.6b.

Consider synchronization of two discrete dynamical systems, as shown in Fig. 6.7, with

$$\mathbf{f}^{(\alpha)}(\mathbf{x}_{k+1}, \mathbf{x}_k, \mathbf{p}^{(\alpha)}) = \mathbf{0} \text{ and } \mathbf{f}^{(\beta)}(\mathbf{y}_{k+1}, \mathbf{y}_k, \mathbf{p}^{(\beta)}) = \mathbf{0}. \tag{6.67}$$

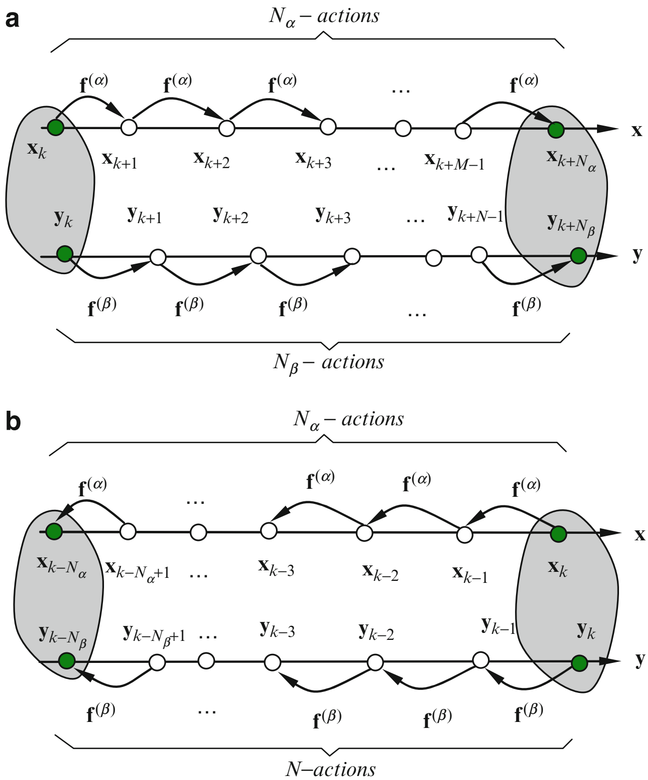

Fig. 6.6 Companion of two discrete nonlinear systems: (**a**) positive companion, (**b**) negative companion

For the initial state, there is a relation as

$$\boldsymbol{\varphi}(\mathbf{x}_k, \mathbf{y}_k, \boldsymbol{\lambda}) = \mathbf{0}. \tag{6.68}$$

For the positive synchronization, there are N_α-actions with function $\mathbf{f}^{(\alpha)}$ and mapping $P_{\alpha+}$ and N_β-actions with function $\mathbf{f}^{(\beta)}$ and mapping $P_{\beta+}$

$$\begin{aligned} &\mathbf{f}^{(\alpha)}(\mathbf{x}_{k+i}, \mathbf{x}_{k+i-1}, \mathbf{p}^{(\alpha)}) = \mathbf{0} \text{ for } i = 1, 2, \cdots, N_\alpha, \\ &\mathbf{f}^{(\beta)}(\mathbf{y}_{k+j}, \mathbf{y}_{k+j-1}, \mathbf{p}^{(\beta)}) = \mathbf{0} \text{ for } j = 1, 2, \cdots, N_\beta; \end{aligned} \tag{6.69}$$

and the synchronization is based on

$$\boldsymbol{\varphi}(\mathbf{x}_{k+N_\alpha}, \mathbf{y}_{k+N_\beta}, \boldsymbol{\lambda}) = \mathbf{0}. \tag{6.70}$$

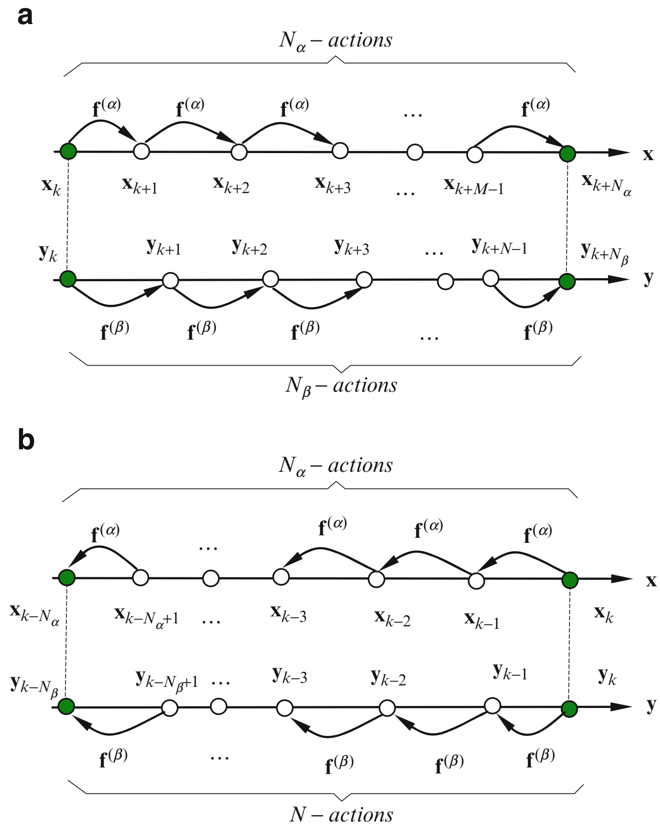

Fig. 6.7 Synchronization of two discrete nonlinear systems: (**a**) positive synchronization, (**b**) negative synchronization

For the negative synchronization, there are N_α-actions with function $\mathbf{f}^{(\alpha)}$ and mapping $P_{\alpha-}$ and N_β-actions with function $\mathbf{f}^{(\beta)}$ and mapping $P_{\beta-}$

$$\begin{aligned}&\mathbf{f}^{(\alpha)}(\mathbf{x}_{k-i+1},\mathbf{x}_{k-i-1},\mathbf{p}^{(\alpha)})=\mathbf{0}\text{ for }i=1,2,\cdots,N_\alpha,\\&\mathbf{f}^{(\beta)}(\mathbf{y}_{k-j},\mathbf{y}_{k-j-1},\mathbf{p}^{(\beta)})=\mathbf{0}\text{ for }j=1,2,\cdots,N_\beta\end{aligned}\tag{6.71}$$

and the synchronization is based on

$$\boldsymbol{\varphi}(\mathbf{x}_{k-N_\alpha},\mathbf{y}_{k-N_\beta},\boldsymbol{\lambda})=\mathbf{0}.\tag{6.72}$$

Thus there is a relation

$$\mathbf{x}_k=P_{\boldsymbol{\varphi}-}\circ P_{\beta-}^{(N_\beta)}\circ P_{\boldsymbol{\varphi}+}\circ P_{\alpha+}^{(N_\alpha)}\mathbf{x}_k,\tag{6.73}$$

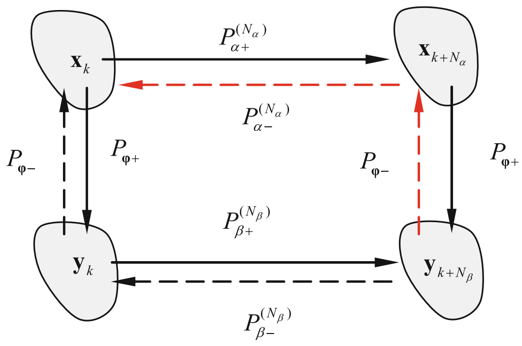

Fig. 6.8 Commutative mapping diagram for synchronization

where

$$
\begin{aligned}
&P_{\boldsymbol{\varphi}-} \circ P_{\beta-}^{(N_\beta)} \circ P_{\boldsymbol{\varphi}+} \circ P_{\alpha+}^{(N_\alpha)} \\
&\quad = P_{\boldsymbol{\varphi}-} \circ \underbrace{P_{\beta-} \circ P_{\beta-} \circ \cdots \circ P_{\beta-}}_{N_\beta-actions} \circ P_{\boldsymbol{\varphi}+} \circ \underbrace{P_{\alpha+} \circ P_{\alpha+} \circ \cdots \circ P_{\alpha+}}_{N_\alpha-actions}.
\end{aligned} \tag{6.74}
$$

From Eq. (6.73), we have

$$
\begin{aligned}
&\mathbf{x}_{k+N_\alpha} = P_{\alpha+}^{(N_\alpha)}\mathbf{x}_k \text{ and } \mathbf{y}_{k+N_\beta} = P_{\boldsymbol{\varphi}+}\mathbf{x}_{k+N_\alpha}, \\
&\mathbf{y}_k = P_{\beta-}^{(N_\beta)}\mathbf{y}_{k+N_\beta} \text{ and } \mathbf{x}_k = P_{\boldsymbol{\varphi}-}\mathbf{y}_k
\end{aligned} \tag{6.75}
$$

and

$$
\begin{aligned}
&\mathbf{x}_{k+N_\alpha} = P_{\alpha+}^{(N_\alpha)}\mathbf{x}_k \text{ and } \mathbf{y}_{k+N_\beta} = P_{\boldsymbol{\varphi}+}\mathbf{x}_{k+N_\alpha}, \\
&P_{\boldsymbol{\varphi}+}\mathbf{x}_k = \mathbf{y}_k \text{ and } P_{\beta+}^{(N_\beta)}\mathbf{y}_k = \mathbf{y}_{k+N_\beta}.
\end{aligned} \tag{6.76}
$$

The corresponding commutative diagram is given in Fig. 6.8. The solid and dashed arrows give the positive and negative mappings, respectively.

From the above discussion on synchronization of $P_{\alpha+}^{(N_\alpha)}$ and $P_{\beta+}^{(N_\beta)}$ under the constraint $\boldsymbol{\varphi}$, the following relations should exist

$$
\begin{aligned}
&\mathbf{x}'_k = P_{\boldsymbol{\varphi}-} \circ P_{\beta-}^{(N_\beta)} \circ P_{\boldsymbol{\varphi}+} \circ P_{\alpha+}^{(N_\alpha)}\mathbf{x}_k, \text{ or} \\
&\mathbf{x}'_k = P_{\alpha-}^{(N_\alpha)} \circ P_{\boldsymbol{\varphi}-} \circ P_{\beta+}^{(N_\beta)} \circ P_{\boldsymbol{\varphi}+}\mathbf{x}_k;
\end{aligned} \tag{6.77}
$$

The above equation forms an iterative mapping. If the fixed point exits, i.e.,

$$
\mathbf{x}'_k = \mathbf{x}_k, \tag{6.78}
$$

then the synchronization of $P_{\alpha+}^{(N_\alpha)}$ and $P_{\beta-}^{(N_\beta)}$ under the constraint $\boldsymbol{\varphi}$ exists

$$\begin{aligned}&\mathbf{x}_{k+N_\alpha} = P_{\alpha+}^{(N_\alpha)}\mathbf{x}_k, \text{ and } \mathbf{y}_{k+N_\beta} = P_{\beta+}^{(N_\beta)}\mathbf{y}_k;\\ &\mathbf{y}_k = P_{\boldsymbol{\varphi}+}\mathbf{x}_k \text{ and } \mathbf{y}_{k+N_\beta} = P_{\boldsymbol{\varphi}+}\mathbf{x}_{k+N_\alpha}.\end{aligned} \tag{6.79}$$

Theorem 6.6 *Consider two discrete dynamical systems $(P_\alpha, \mathbf{f}^{(\alpha)})$ and $(P_\beta, \mathbf{f}^{(\beta)})$ as in Eq. (6.48) with*

$$\begin{aligned}&P_{\alpha+} : \mathbf{x}_k \to \mathbf{x}_{k+1} \text{ and } P_{\alpha-} : \mathbf{x}_{k+1} \to \mathbf{x}_k,\\ &\mathbf{f}^{(\alpha)}(\mathbf{x}_k, \mathbf{x}_{k+1}, \mathbf{p}^{(\alpha)}) = \mathbf{0};\end{aligned} \tag{6.80}$$

and

$$\begin{aligned}&P_{\beta+} : \mathbf{y}_k \to \mathbf{y}_{k+1} \text{ and } P_{\alpha-} : \mathbf{y}_{k+1} \to \mathbf{y}_k,\\ &\mathbf{f}^{(\beta)}(\mathbf{y}_k, \mathbf{y}_{k+1}, \mathbf{p}^{(\beta)}) = \mathbf{0}.\end{aligned} \tag{6.81}$$

For two points $\mathbf{x}_k \in D_\alpha$ and $\mathbf{y}_k \in D_\beta$, there is a specific, differentiable, vector function $\boldsymbol{\varphi} = (\varphi_1, \varphi_2, \cdots, \varphi_l)^{\mathrm{T}} \in \mathscr{R}^l$. The synchronization of two discrete dynamical systems $(P_\alpha, \mathbf{f}^{(\alpha)})$ and $(P_\beta, \mathbf{f}^{(\beta)})$ is under the following constraints

$$\boldsymbol{\varphi}(\mathbf{x}_k, \mathbf{y}_k, \boldsymbol{\lambda}) = \mathbf{0} \text{ and } \boldsymbol{\varphi}(\mathbf{x}_{k+1}, \mathbf{y}_{k+1}, \boldsymbol{\lambda}) = \mathbf{0}. \tag{6.82}$$

Consider a resultant hybrid mapping relation as

$$\mathbf{x}'_k = P\mathbf{x}_k = P_{\boldsymbol{\varphi}-} \circ P_{\beta-} \circ P_{\boldsymbol{\varphi}+} \circ P_{\alpha+}\mathbf{x}_k \tag{6.83}$$

with

$$\begin{aligned}&P_{\alpha+} : \mathbf{x}_k \to \mathbf{x}_{k+1} \text{ with } \mathbf{f}^{(\alpha)}(\mathbf{x}_{k+1}, \mathbf{x}_k, \mathbf{p}^{(\alpha)}) = \mathbf{0},\\ &P_{\boldsymbol{\varphi}+} : \mathbf{x}_{k+1} \to \mathbf{y}_{k+1} \text{ with } \boldsymbol{\varphi}(\mathbf{x}_{k+1}, \mathbf{y}_{k+1}, \boldsymbol{\lambda}) = \mathbf{0},\\ &P_{\beta-} : \mathbf{y}_{k+1} \to \mathbf{y}_k \text{ with } \mathbf{f}^{(\beta)}(\mathbf{y}_{k+1}, \mathbf{y}_k, \mathbf{p}^{(\alpha)}) = \mathbf{0},\\ &P_{\boldsymbol{\varphi}-} : \mathbf{y}_k \to \mathbf{x}'_k \text{ with } \boldsymbol{\varphi}(\mathbf{x}'_k, \mathbf{y}_k, \boldsymbol{\lambda}) = \mathbf{0};\\ &\mathbf{x}'_k = \mathbf{x}_k\end{aligned} \tag{6.84}$$

and

$$DP(\mathbf{x}_k^*) = DP_{\boldsymbol{\varphi}-}(\mathbf{y}_k^*) \cdot DP_{\beta-}(\mathbf{y}_{k+1}^*) \cdot DP_{\boldsymbol{\varphi}+}(\mathbf{x}_{k+1}^*) \cdot DP_{\alpha+}(\mathbf{x}_k^*), \tag{6.85}$$

where

$$\begin{aligned}&DP(\mathbf{x}_k^*) = \left[\frac{\partial \mathbf{x}'_k}{\partial \mathbf{x}_k}\right]_{\mathbf{x}_k^*}, DP_{\alpha+}(\mathbf{x}_k^*) = \left[\frac{\partial \mathbf{x}_{k+1}}{\partial \mathbf{x}_k}\right]_{\mathbf{x}_k^*,} , DP_{\boldsymbol{\varphi}+}(\mathbf{x}_{k+1}^*) = \left[\frac{\partial \mathbf{y}_{k+1}}{\partial \mathbf{x}_{k+1}}\right]_{\mathbf{x}_{k+1}^*},\\ &DP_{\beta-}(\mathbf{y}_{k+1}^*) = \left[\frac{\partial \mathbf{y}_k}{\partial \mathbf{y}_{k+1}}\right]_{\mathbf{y}_{k+1}^*}, DP_{\boldsymbol{\varphi}-}(\mathbf{y}_k^*) = \left[\frac{\partial \mathbf{x}'_k}{\partial \mathbf{y}_k}\right]_{\mathbf{y}_k^*}.\end{aligned} \tag{6.86}$$

(i) *The* $(1:1)$ *synchronization of two discrete dynamical systems of* $(P_\alpha, \mathbf{f}^{(\alpha)})$ *and* $(P_\beta, \mathbf{f}^{(\beta)})$ *is persistent if and only if all the eigenvalues* λ_i $(i = 1, 2, \cdots, n)$ *of* $DP(\mathbf{x}_k^*)$ *lie in the unit circles, i.e.,*

$$|\lambda_i| < 1 \text{ for } i = 1, 2, \cdots, n. \tag{6.87}$$

(ii) *The* $(1:1)$ *synchronization of two discrete dynamical systems of* $(P_\alpha, \mathbf{f}^{(\alpha)})$ *and* $(P_\beta, \mathbf{f}^{(\beta)})$ *is a saddle-node vanishing if and only if at least one of the real eigenvalues* λ_i *(*$i = 1, 2, \cdots, n_1$ *and* $n_1 \le n$*) of* $DP(\mathbf{x}_k^*)$ *is positive one (+1) and the other eigenvalues are in the unit circle, i.e.,*

$$\lambda_i = 1 \text{ and } |\lambda_j| < 1 \text{ for } i, j \in \{1, 2, \cdots, n\} \text{ and } j \ne i. \tag{6.88}$$

(iii) *The* $(1:1)$ *synchronization of two discrete dynamical systems of* $(P_\alpha, \mathbf{f}^{(\alpha)})$ *and* $(P_\beta, \mathbf{f}^{(\beta)})$ *is a period-doubling vanishing if and only if at least one of the real eigenvalues* λ_i *(*$i = 1, 2, \cdots, n_1$ *and* $n_1 \le n$*) of* $DP(\mathbf{x}_k^*)$ *is negative one (−1) and the other eigenvalues are in the unit circle, i.e.,*

$$\lambda_i = -1 \text{ and } |\lambda_j| < 1 \text{ for } i, j \in \{1, 2, \cdots, n\} \text{ and } j \ne i. \tag{6.89}$$

(iv) *The* $(1:1)$ *synchronization of two discrete dynamical systems of* $(P_\alpha, \mathbf{f}^{(\alpha)})$ *and* $(P_\beta, \mathbf{f}^{(\beta)})$ *is a Naimark vanishing if and only if one pair of all the complex eigenvalues* $\lambda_i = \alpha_i \pm \beta_i \mathbf{i}$ *(*$i = 1, 2, \cdots, n_1$ *and* $n_1 \le n/2$*) of* $DP(\mathbf{x}_k^*)$ *are on the unit circle and the other eigenvalues are in the unit circle, i.e.,*

$$|\lambda_i| = \sqrt{\alpha_i^2 + \beta_i^2} = 1 \text{ and } |\lambda_j| < 1 \text{ for } i, j \in \{1, 2, \cdots, n\} \text{ and } j \ne i. \tag{6.90}$$

(v) *The* $(1:1)$ *synchronization of two discrete dynamical systems of* $(P_\alpha, \mathbf{f}^{(\alpha)})$ *and* $(P_\beta, \mathbf{f}^{(\beta)})$ *is an (*$l_1 : l_2 : l_3$*) vanishing if and only if* l_1 *and* l_2 *real eigenvalues* λ_i *of* $DP(\mathbf{x}_k^*)$ *are (−1) and (+1), respectively, and* l_3*-pairs of complex eigenvalues* $\lambda_i = \alpha_i \pm \beta_i \mathbf{i}$ *(*$i = 1, 2, \cdots, n_1$ *and* $n_1 \le n/2$*) of* $DP(\mathbf{x}_k^*)$ *are on the unit circle and the other eigenvalues are in the unit circle, i.e.,*

$$\begin{aligned}
&\lambda_i = -1 \text{ for } i = i_1, i_2, \cdots, i_{l_1} \in \{1, 2, \cdots, n\} \\
&\lambda_j = +1 \text{ for } j = j_1, j_2, \cdots, j_{l_2} \in \{1, 2, \cdots, n\} \\
&|\lambda_r| = \sqrt{\alpha_r^2 + \beta_r^2} = 1 \text{ for } r = r_1, r_2, \cdots, r_{l_3} \in \{1, 2, \cdots, n\} \\
&|\lambda_s| < 1 \text{ for } s \in \{1, 2, \cdots, n\} \text{ and } s \notin \{i, j, r\}.
\end{aligned} \tag{6.91}$$

(vi) *The* $(1:1)$ *synchronization of two discrete dynamical systems of* $(P_\alpha, \mathbf{f}^{(\alpha)})$ *and* $(P_\beta, \mathbf{f}^{(\beta)})$ *is instantaneous if and only if at least one of the eigenvalues* $\lambda_i(i = 1, 2, \ldots, n)$ *of* $DP(\mathbf{x}_k^*)$ *lies out of the unit circle, i.e.,*

$$|\lambda_i| > 1 \text{ for } i \in \{1, 2, \cdots, n\}. \tag{6.92}$$

Proof The proof can be referred to Luo [2]. □

Theorem 6.7 *Consider two discrete dynamical systems* $(P_\alpha, \mathbf{f}^{(\alpha)})$ *and* $(P_\beta, \mathbf{f}^{(\beta)})$ *as in Eq. (6.48) with*

$$\begin{aligned} &P_{\alpha+} : \mathbf{x}_k \to \mathbf{x}_{k+1} \text{ and } P_{\alpha-} : \mathbf{x}_{k+1} \to \mathbf{x}_k \\ &\mathbf{f}^{(\alpha)}(\mathbf{x}_k, \mathbf{x}_{k+1}, \mathbf{p}^{(\alpha)}) = \mathbf{0}, \end{aligned} \tag{6.93}$$

and

$$\begin{aligned} &P_{\beta+} : \mathbf{y}_k \to \mathbf{y}_{k+1} \text{ and } P_{\beta-} : \mathbf{y}_{k+1} \to \mathbf{y}_k, \\ &\mathbf{f}^{(\beta)}(\mathbf{y}_k, \mathbf{y}_{k+1}, \mathbf{p}^{(\beta)}) = \mathbf{0}. \end{aligned} \tag{6.94}$$

For two points $\mathbf{x}_k \in D_\alpha$ *and* $\mathbf{y}_k \in D_\beta$*, there is a specific, differentiable, vector function* $\boldsymbol{\varphi} = (\varphi_1, \varphi_2, \cdots, \varphi_l)^{\mathrm{T}} \in \mathscr{R}^l$*. The* $(N_\alpha : N_\beta)$*-synchronization of two discrete dynamical systems* $(P_\alpha, \mathbf{f}^{(\alpha)})$ *and* $(P_\beta, \mathbf{f}^{(\beta)})$ *is under the following constraints*

$$\boldsymbol{\varphi}(\mathbf{x}_k, \mathbf{y}_k, \boldsymbol{\lambda}) = \mathbf{0} \text{ and } \boldsymbol{\varphi}(\mathbf{x}_{k+N_\alpha}, \mathbf{y}_{k+N_\beta}, \boldsymbol{\lambda}) = \mathbf{0}. \tag{6.95}$$

Consider a resultant hybrid mapping relation as

$$\mathbf{x}'_k = P^{(N_\alpha : N_\beta)}\mathbf{x}_k = P_{\boldsymbol{\varphi}-} \circ P^{(N_\beta)}_{\beta-} \circ P_{\boldsymbol{\varphi}+} \circ P^{(N_\alpha)}_{\alpha+}\mathbf{x}_k \tag{6.96}$$

with

$$\begin{aligned} &P^{(N_\alpha)}_{\alpha+} : \mathbf{x}_k \to \mathbf{x}_{k+N_\alpha} \text{with} \\ &\left.\begin{aligned} &\mathbf{f}^{(\alpha)}(\mathbf{x}_{k+1}, \mathbf{x}_k, \mathbf{p}^{(\alpha)}) = \mathbf{0}, \\ &\mathbf{f}^{(\alpha)}(\mathbf{x}_{k+2}, \mathbf{x}_{k+1}, \mathbf{p}^{(\alpha)}) = \mathbf{0}, \\ &\vdots \\ &\mathbf{f}^{(\alpha)}(\mathbf{x}_{k+N_\alpha}, \mathbf{x}_{k+N_\alpha-1}, \mathbf{p}^{(\alpha)}) = \mathbf{0}, \end{aligned}\right\} \\ &P_{\boldsymbol{\varphi}+} : \mathbf{x}_{k+1} \to \mathbf{y}_{k+1} \text{ with } \boldsymbol{\varphi}(\mathbf{x}_{k+N_\alpha}, \mathbf{y}_{k+N_\beta}, \boldsymbol{\lambda}) = \mathbf{0}; \\ &P^{(N_\beta)}_{\beta-} : \mathbf{y}_{k+N_\beta} \to \mathbf{y}_k \text{with} \\ &\left.\begin{aligned} &\mathbf{f}^{(\beta)}(\mathbf{y}_{k+N_\beta}, \mathbf{y}_{k+N_\beta-1}, \mathbf{p}^{(\beta)}) = \mathbf{0}, \\ &\vdots \\ &\mathbf{f}^{(\beta)}(\mathbf{y}_{k+2}, \mathbf{y}_{k+1}, \mathbf{p}^{(\beta)}) = \mathbf{0}, \\ &\mathbf{f}^{(\beta)}(\mathbf{y}_{k+1}, \mathbf{y}_k, \mathbf{p}^{(\beta)}) = \mathbf{0}, \end{aligned}\right\} \\ &P_{\boldsymbol{\varphi}-} : \mathbf{y}_k \to \mathbf{x}'_k \text{ with } \boldsymbol{\varphi}(\mathbf{x}'_k, \mathbf{y}_k, \boldsymbol{\lambda}) = \mathbf{0}; \\ &\mathbf{x}'_k = \mathbf{x}_k \end{aligned} \tag{6.97}$$

and

$$DP^{(N_\alpha:N_\beta)}(\mathbf{x}_k^*) = DP_{\boldsymbol{\varphi}-}(\mathbf{y}_k^*) \cdot DP_{\beta-}^{(N_\beta)}(\mathbf{y}_{k+N_\alpha}^*) \cdot DP_{\boldsymbol{\varphi}+}(\mathbf{x}_{k+N_\alpha}^*) \cdot DP_{\alpha+}^{(N_\alpha)}(\mathbf{x}_k^*), \quad (6.98)$$

where

$$\begin{aligned} &DP_{\alpha+}^{(N_\alpha)}(\mathbf{x}_k^*) = DP_{\alpha+}(\mathbf{x}_{k+N_\alpha-1}^*) \cdot \ldots \cdot DP_{\alpha+}(\mathbf{x}_{k+1}^*) \cdot DP_{\alpha+}(\mathbf{x}_k^*), \\ &DP_{\beta-}^{(N_\beta)}(\mathbf{y}_{k+N_\beta}^*) = DP_{\beta-}(\mathbf{x}_{k+1}^*) \cdot \ldots \cdot DP_{\beta-}(\mathbf{x}_{k+N_\beta-1}^*) \cdot DP_{\beta-}(\mathbf{x}_{k+N_\beta}^*), \end{aligned} \quad (6.99)$$

$$\begin{aligned} &DP^{(N_\alpha:N_\beta)}(\mathbf{x}_k^*) = \left[\frac{\partial \mathbf{x}'_k}{\partial \mathbf{x}_k}\right]_{\mathbf{x}_k^*}, \\ &DP_{\alpha+}^{(N_\alpha)}(\mathbf{x}_k^*) = \prod\nolimits_{j=N_\alpha}^{1} \left[\frac{\partial \mathbf{x}_{k+j}}{\partial \mathbf{x}_{k+j-1}}\right]_{\mathbf{x}_{k+j-1}^*}, DP_{\boldsymbol{\varphi}+}(\mathbf{x}_{k+N_\alpha}^*) = \left[\frac{\partial \mathbf{y}_{k+N_\beta}}{\partial \mathbf{x}_{k+N_\alpha}}\right]_{\mathbf{x}_{k+N_\alpha}^*}, \\ &DP_{\beta-}^{(N_\beta)}(\mathbf{y}_{k+N_\beta}^*) = \prod\nolimits_{j=1}^{N_\beta} \left[\frac{\partial \mathbf{y}_{k+N_\beta-j}}{\partial \mathbf{y}_{k+N_\beta-j+1}}\right]_{\mathbf{y}_{k+N_\beta-j+1}^*}, DP_{\boldsymbol{\varphi}-}(\mathbf{y}_k^*) = \left[\frac{\partial \mathbf{x}'_k}{\partial \mathbf{y}_k}\right]_{\mathbf{y}_k^*}. \end{aligned} \quad (6.100)$$

(i) *The* $(N_\alpha : N_\beta)$*-synchronization of two discrete dynamical systems of* $(P_\alpha, \mathbf{f}^{(\alpha)})$ *and* $(P_\beta, \mathbf{f}^{(\beta)})$ *is persistent if and only if all the eigenvalues* λ_i *(*$i = 1, 2, \cdots, n$*) of* $DP^{(N_\alpha:N_\beta)}(\mathbf{x}_k^*)$ *lie in the unit circles, i.e.,*

$$|\lambda_i| < 1 \text{ for } i = 1, 2, \cdots, n. \quad (6.101)$$

(ii) *The* $(N_\alpha : N_\beta)$*-synchronization of two discrete dynamical systems of* $(P_\alpha, \mathbf{f}^{(\alpha)})$ *and* $(P_\beta, \mathbf{f}^{(\beta)})$ *is a saddle-node vanishing if and only if at least one of the real eigenvalues* λ_i *(*$i = 1, 2, \cdots, n_1$ *and* $n_1 \le n$*) of* $DP^{(N_\alpha:N_\beta)}(\mathbf{x}_k^*)$ *is positive one (+1) and the other eigenvalues are in the unit circle, i.e.,*

$$\lambda_i = 1 \text{ and } |\lambda_j| < 1 \text{ for } i, j \in \{1, 2, \cdots, n\} \text{ and } j \neq i. \quad (6.102)$$

(iii) *The* $(N_\alpha : N_\beta)$ *synchronization of two discrete dynamical systems of* $(P_\alpha, \mathbf{f}^{(\alpha)})$ *and* $(P_\beta, \mathbf{f}^{(\beta)})$ *is a period-doubling vanishing if and only if at least one of the real eigenvalues* λ_i *(*$i = 1, 2, \cdots, n_1$ *and* $n_1 \le n$*) of* $DP^{(N_\alpha:N_\beta)}(\mathbf{x}_k^*)$ *is negative one (−1) and the other eigenvalues are in the unit circle, i.e.,*

$$\lambda_i = -1 \text{ and } |\lambda_j| < 1 \text{ for } i, j \in \{1, 2, \cdots, n\} \text{ and } j \neq i. \quad (6.103)$$

(iv) *The* $(N_\alpha : N_\beta)$*-synchronization of two discrete dynamical systems of* $(P_\alpha, \mathbf{f}^{(\alpha)})$ *and* $(P_\beta, \mathbf{f}^{(\beta)})$ *is a Naimark vanishing if and only if at least one pair of all* *the complex eigenvalues* $\lambda_i = \alpha_i \pm \beta_i \mathbf{i}$ *(* $i = 1, 2, \cdots, n_1$ *and* $n_1 \le n/2$ *) of* $DP^{(N_\alpha : N_\beta)}(\mathbf{x}_k^*)$ *are on the unit circle and the other eigenvalues are in the unit circle, i.e.,*

$$|\lambda_i| = \sqrt{\alpha_i^2 + \beta_i^2} = 1 \text{ and } |\lambda_j| < 1 \text{ for } i, j \in \{1, 2, \cdots, n\} \text{ and } j \ne i. \tag{6.104}$$

(v) *The* $(N_\alpha : N_\beta)$ *synchronization of two discrete dynamical systems of* $(P_\alpha, \mathbf{f}^{(\alpha)})$ *and* $(P_\beta, \mathbf{f}^{(\beta)})$ *is an* ($l_1 : l_2 : l_3$) *vanishing if and only if* l_1 *and* l_2 *real eigenvalues* λ_i *of* $DP^{(N_\alpha : N_\beta)}(\mathbf{x}_k^*)$ *are (−1) and (+1), respectively, and* l_3*-pairs of complex eigenvalues* $\lambda_i = \alpha_i \pm \beta_i \mathbf{i}$ *(*$i = 1, 2, \cdots, n_1$ *and* $n_1 \le n/2$*) of* $DP^{(N_\alpha : N_\beta)}(\mathbf{x}_k^*)$ *are on the unit circle and the other eigenvalues are in the unit circle, i.e.,*

$$\begin{aligned} \lambda_i &= -1 \text{ for } i = i_1, i_2, \cdots, i_{l_1} \in \{1, 2, \cdots, n\}, \\ \lambda_j &= +1 \text{ for } j = j_1, j_2, \cdots, j_{l_2} \in \{1, 2, \cdots, n\}, \\ |\lambda_r| &= \sqrt{\alpha_r^2 + \beta_r^2} = 1 \text{ for } r = r_1, r_2, \cdots, r_{l_3} \in \{1, 2, \cdots, n\}, \\ |\lambda_s| &< 1 \text{ for } s \in \{1, 2, \cdots, n\} \text{ and } s \notin \{i, j, r\}. \end{aligned} \tag{6.105}$$

(vi) *The* $(N_\alpha : N_\beta)$ *synchronization of two discrete dynamical systems of* $(P_\alpha, \mathbf{f}^{(\alpha)})$ *and* $(P_\beta, \mathbf{f}^{(\beta)})$ *is instantaneous if and only if at least one of the eigenvalues* λ_i $(i = 1, 2, \cdots, n)$ *of* $DP^{(N_\alpha : N_\beta)}(\mathbf{x}_k^*)$ *lies out of the unit circle, i.e.,*

$$|\lambda_i| > 1 \text{ for } i \in \{1, 2, \cdots, n\}. \tag{6.106}$$

Proof The proof can be referred to Luo [2]. □

Fixed points in nonlinear discrete dynamical systems possess many types of unstable states from eigenvalue analysis. From the similar ideas, the instantaneous $(N_\alpha : N_\beta)$ synchronization of two discrete dynamical systems can be classified. Therefore, such instantaneous synchronization classification will not be presented herein. If $N_\alpha \to \infty$ and $N_\beta \to \infty$, the $(N_\alpha : N_\beta)$ synchronization of two discrete dynamical systems should be chaotic. Consider two hybrid maps

$$\begin{aligned} P_{+}^{(\Sigma_{i=1}^{n} N_\beta^i \oplus N_\alpha^i)} &= \underbrace{P_{\beta+}^{(N_\beta^n)} \circ P_{\alpha+}^{(N_\alpha^n)} \circ \cdots \circ P_{\beta+}^{(N_\beta^1)} \circ P_{\alpha+}^{(N_\alpha^1)}}_{n-terms}, \\ P_{+}^{(\Sigma_{j=1}^{m} M_\beta^j \oplus M_\alpha^j)} &= \underbrace{P_{\beta+}^{(M_\beta^m)} \circ P_{\alpha+}^{(M_\alpha^m)} \circ \cdots \circ P_{\beta+}^{(M_\beta^1)} \circ P_{\alpha+}^{(M_\alpha^1)}}_{m-terms}. \end{aligned} \tag{6.107}$$

$$\begin{aligned} P_{-}^{(\Sigma_{i=n}^{1} N_\alpha^i \oplus N_\beta^i)} &= \underbrace{P_{\alpha-}^{(N_\alpha^1)} \circ P_{\beta-}^{(N_\beta^1)} \circ \cdots \circ P_{\alpha-}^{(N_\alpha^m)} \circ P_{\beta-}^{(N_\beta^m)}}_{n-terms}, \\ P_{-}^{(\Sigma_{j=m}^{1} M_\alpha^j \oplus M_\beta^j)} &= \underbrace{P_{\alpha-}^{(M_\alpha^1)} \circ P_{\beta-}^{(M_\beta^1)} \circ \cdots \circ P_{\alpha-}^{(M_\alpha^m)} \circ P_{\beta-}^{(M_\beta^m)}}_{m-terms}. \end{aligned} \tag{6.108}$$

The $(N_\beta \oplus N_\alpha : M_\beta \oplus M_\alpha)$-hybrid synchronization of two discrete systems with two maps $P_+^{(\Sigma_{i=1}^n N_\beta^i \oplus N_\alpha^i)}$ and $P_+^{(\Sigma_{j=1}^m M_\beta^j \oplus M_\alpha^j)}$ can be investigated via the following map

$$
\begin{aligned}
P^{(N_\beta \oplus N_\alpha : M_\beta \oplus M_\alpha)} \mathbf{x}_k &= P_{\boldsymbol{\varphi}-} \circ P_-^{(\Sigma_{j=m}^1 M_\alpha^j \oplus M_\beta^j)} \circ P_{\boldsymbol{\varphi}+} \circ P_+^{(\Sigma_{i=1}^n N_\beta^i \oplus N_\alpha^i)} \mathbf{x}_k, \text{ or} \\
P^{(N_\beta \oplus N_\alpha : M_\beta \oplus M_\alpha)} \mathbf{x}_k &= P_{\boldsymbol{\varphi}-} \circ P_-^{(\Sigma_{i=n}^1 N_\alpha^i \oplus N_\beta^i)} \circ P_{\boldsymbol{\varphi}+} \circ P_+^{(\Sigma_{j=1}^m M_\beta^j \oplus M_\alpha^j)} \mathbf{x}_k.
\end{aligned} \tag{6.109}
$$

Thus,

$$
\mathbf{x}'_k = P^{(N_\beta \oplus N_\alpha : M_\beta \oplus M_\alpha)} \mathbf{x}_k. \tag{6.110}
$$

Similar to the $(N_\alpha : N_\beta)$-synchronization in Theorem 6.7, the corresponding fixed point and the stability conditions of Eq. (6.110) gives the $(N_\beta \oplus N_\alpha : M_\beta \oplus M_\alpha)$-hybrid synchronization of two discrete systems. This concept can be extended to the discrete dynamical systems with multiple maps.

As in discrete dynamical systems with multiple maps in Section 6.2, the synchronization for the resultant mappings in multiple different maps can be developed.

Definition 6.14 Consider two sets of discrete dynamical systems $\cup_{i=1}(P_{\alpha_i}, \mathbf{f}^{(\alpha_i)})$ and $\cup_{j=1}(P_{\beta_j}, \mathbf{f}^{(\beta_j)})$ as in Eq. (6.48) for each discrete system with

$$
\begin{aligned}
&P_{\alpha_i+} : \mathbf{x}_k \to \mathbf{x}_{k+1} \text{ and } P_{\alpha_i-} : \mathbf{x}_{k+1} \to \mathbf{x}_k, \\
&\mathbf{f}^{(\alpha_i)}(\mathbf{x}_k, \mathbf{x}_{k+1}, \mathbf{p}^{(\alpha_i)}) = \mathbf{0},
\end{aligned} \tag{6.111}
$$

and

$$
\begin{aligned}
&P_{\beta_j+} : \mathbf{y}_k \to \mathbf{y}_{k+1} \text{ and } P_{\beta_j-} : \mathbf{y}_{k+1} \to \mathbf{y}_k, \\
&\mathbf{f}^{(\beta_j)}(\mathbf{y}_k, \mathbf{y}_{k+1}, \mathbf{p}^{(\beta_j)}) = \mathbf{0}.
\end{aligned} \tag{6.112}
$$

For the two sets of discrete dynamical systems, the resultant mappings are

$$
\begin{aligned}
P^+_{(N_{\alpha_m} \cdots N_{\alpha_2} N_{\alpha_1})} &= \underbrace{P^+_{\alpha_m^{N_{\alpha m}}} \circ \cdots \circ P^+_{\alpha_2^{N_{\alpha 2}}} \circ P^+_{\alpha_1^{N_{\alpha 1}}}}_{m-terms}, \\
P^-_{(N_{\alpha_1} N_{\alpha_2} \cdots N_{\alpha_m})} &= \underbrace{P^-_{\alpha_1^{N_{\alpha 1}}} \circ P^-_{\alpha_2^{N_{\alpha 2}}} \circ \cdots \circ P^-_{\alpha_m^{N_{\alpha m}}}}_{m-terms};
\end{aligned} \tag{6.113}
$$

and

$$
\begin{aligned}
P^+_{(N_{\beta_n} \cdots N_{\beta_2} N_{\beta_1})} &= \underbrace{P^+_{\beta_n^{N_{\beta n}}} \circ \cdots \circ P^+_{\beta_2^{N_{\beta 2}}} \circ P^+_{\beta_1^{N_{\beta 1}}}}_{n-terms}, \\
P^-_{(N_{\beta_1} N_{\beta_2} \cdots N_{\beta_m})} &= \underbrace{P^-_{\beta_1^{N_{\beta 1}}} \circ P^-_{\beta_2^{N_{\beta 2}}} \circ \cdots \circ P^-_{\beta_n^{N_{\beta n}}}}_{n-terms},
\end{aligned} \tag{6.114}
$$

where

$$N_\alpha = \sum\nolimits_{i=1}^{m} N_{\alpha_i} \text{ and } N_\beta = \sum\nolimits_{j=1}^{n} N_{\beta_j}. \tag{6.115}$$

For two points $\mathbf{x}_k \in D_{\alpha_1}$ and $\mathbf{y}_k \in D_{\beta_1}$, there is a specific, differentiable, vector function $\boldsymbol{\varphi} = (\varphi_1, \varphi_2, \cdots, \varphi_l)^{\mathrm{T}} \in \mathscr{R}^l$.

(i) If

$$\boldsymbol{\varphi}(\mathbf{x}_{k+N_\alpha}^{(\alpha_m)}, \mathbf{x}_{k+N_\beta}^{(\beta_n)}, \boldsymbol{\lambda}) = \mathbf{0}, \tag{6.116}$$

then the two discrete dynamical systems $\cup_{i=1}(P_{\alpha_i}, \mathbf{f}^{(\alpha_i)})$ and $\cup_{j=1}(P_{\beta_j}, \mathbf{f}^{(\beta_j)})$ are called the $(N_\alpha : N_\beta)$-synchronization in sense of $\boldsymbol{\varphi}$.

(ii) If

$$\begin{aligned}
&\boldsymbol{\varphi}(\mathbf{x}_{k+N_\alpha}^{(\alpha_m)}, \mathbf{x}_{k+N_\beta}^{(\beta_n)}, \boldsymbol{\lambda}) = \mathbf{0} \text{ with} \\
&\Delta\mathbf{I}^{(\alpha_m\alpha_1)} : \mathbf{x}_{k+N_\alpha(-)}^{(\alpha_m)} \to \mathbf{x}_{k+N_\alpha(+)}^{(\alpha_m)} \text{ and } \Delta\mathbf{I}^{(\beta_n\beta_1)} : \mathbf{x}_{k+N_\beta(-)}^{(\beta_n)} \to \mathbf{x}_{k+N_\beta(+)}^{(\beta_n)}, \\
&\mathbf{x}_{k+N_\alpha(+)}^{(\alpha_m)} = \mathbf{x}_{k+N_\alpha(-)}^{(\alpha_m)} + \Delta\mathbf{I}^{(\alpha_m\alpha_1)} \text{ and } \mathbf{x}_{k+N_\beta(+)}^{(\beta_n)} = \mathbf{x}_{k+N_\beta(-)}^{(\beta_n)} + \Delta\mathbf{I}^{(\beta_n\beta_1)}, \\
&\mathbf{x}_{k+N_\beta(+)}^{(\alpha_m)} = \mathbf{x}_k^{(\alpha_1)} \text{ and } \mathbf{x}_{k+N_\beta(+)}^{(\beta_n)} = \mathbf{x}_k^{(\beta_1)}.
\end{aligned} \tag{6.117}$$

then the two discrete dynamical systems $\cup_{i=1}(P_{\alpha_i}, \mathbf{f}^{(\alpha_i)})$ and $\cup_{j=1}(P_{\beta_j}, \mathbf{f}^{(\beta_j)})$ are called the repeatable $(N_\alpha : N_\beta)$-synchronization in sense of $\boldsymbol{\varphi}$.

The corresponding theorem can be presented as in Theorem 6.7. For convenience, the statement is given as follows.

Theorem 6.8 *Consider two sets of discrete dynamical systems* $\cup_{i=1}(P_{\alpha_i}, \mathbf{f}^{(\alpha_i)})$ *and* $\cup_{j=1}(P_{\beta_j}, \mathbf{f}^{(\beta_j)})$ *as in Eq. (6.48) for each discrete system with*

$$\begin{aligned}
&P_{\alpha_i+} : \mathbf{x}_k \to \mathbf{x}_{k+1} \text{ and } P_{\alpha_i-} : \mathbf{x}_{k+1} \to \mathbf{x}_k, \\
&\mathbf{f}^{(\alpha_i)}(\mathbf{x}_k, \mathbf{x}_{k+1}, \mathbf{p}^{(\alpha_i)}) = \mathbf{0}
\end{aligned} \tag{6.118}$$

and

$$\begin{aligned}
&P_{\beta_j+} : \mathbf{y}_k \to \mathbf{y}_{k+1} \text{ and } P_{\beta_j-} : \mathbf{y}_{k+1} \to \mathbf{y}_k, \\
&\mathbf{f}^{(\beta_j)}(\mathbf{y}_k, \mathbf{y}_{k+1}, \mathbf{p}^{(\beta_j)}) = \mathbf{0}.
\end{aligned} \tag{6.119}$$

For two points $\mathbf{x}_k \in D_{\alpha_1}$ *and* $\mathbf{y}_k \in D_{\beta_1}$, *there is a specific, differentiable, vector function* $\boldsymbol{\varphi} = (\varphi_1, \varphi_2, \cdots, \varphi_l)^{\mathrm{T}} \in \mathscr{R}^l$. *The* $(N_\alpha : N_\beta)$*-synchronization of two sets of discrete dynamical systems* $\cup_{i=1}(P_{\alpha_i}, \mathbf{f}^{(\alpha_i)})$ *and* $\cup_{j=1}(P_{\beta_j}, \mathbf{f}^{(\beta_j)})$ *is under the following constraints*

$$\boldsymbol{\varphi}(\mathbf{x}_k, \mathbf{y}_k, \boldsymbol{\lambda}) = \mathbf{0} \text{ and } \boldsymbol{\varphi}(\mathbf{x}_{k+N_\alpha}, \mathbf{y}_{k+N_\beta}, \boldsymbol{\lambda}) = \mathbf{0}. \tag{6.120}$$

Consider a resultant hybrid mapping relation as

$$\mathbf{x}'_k = P^{(N_\alpha:N_\beta)}\mathbf{x}_k = P_{\boldsymbol{\varphi}-} \circ P_{N_\beta-} \circ P_{\boldsymbol{\varphi}+} \circ P_{N_\alpha+}\mathbf{x}_k \tag{6.121}$$

with

$$P_{N_\alpha+} = P^{+}_{(N_{\alpha_m}\cdots N_{\alpha_2}N_{\alpha_1})} \text{ and } P_{N_\beta-} = P^{-}_{(N_{\beta_1}N_{\beta_2}\cdots N_{\beta_n})}, \tag{6.122}$$

$$\begin{array}{l}
P_{N_{\alpha_i}+} : \mathbf{x}_{k+\Sigma_{r=1}^{i-1}N_{\alpha r}} \to \mathbf{x}_{k+\Sigma_{r=1}^{i}N_{\alpha r}} \text{ with} \\
\left.\begin{array}{l}
\mathbf{f}^{(\alpha_i)}(\mathbf{x}_{k+\Sigma_{r=1}^{i-1}N_{\alpha r}+1}, \mathbf{x}_{k+\Sigma_{r=1}^{i-1}N_{\alpha r}}, \mathbf{p}^{(\alpha)}) = \mathbf{0}, \\
\mathbf{f}^{(\alpha_i)}(\mathbf{x}_{k+\Sigma_{r=1}^{i-1}N_{\alpha r}+2}, \mathbf{x}_{k+\Sigma_{r=1}^{i-1}N_{\alpha r}+1}, \mathbf{p}^{(\alpha)}) = \mathbf{0}, \\
\vdots \\
\mathbf{f}^{(\alpha_i)}(\mathbf{x}_{k+\Sigma_{r=1}^{i}N_{\alpha r}}, \mathbf{x}_{k+\Sigma_{r=1}^{i-1}N_{\alpha r}-1}, \mathbf{p}^{(\alpha_i)}) = \mathbf{0}
\end{array}\right\} \\
\text{for } i = 1, 2, \cdots, m \\
P_{\boldsymbol{\varphi}+} : \mathbf{x}_{k+N_\alpha} \to \mathbf{y}_{k+N_\beta} \text{ with } \boldsymbol{\varphi}(\mathbf{x}_{k+N_\alpha}, \mathbf{y}_{k+N_\beta}, \boldsymbol{\lambda}) = \mathbf{0}; \\
P_{N_{\beta_j}} : \mathbf{y}_{k+N_\beta-\Sigma_{r=j}^{n}N_{\beta_r}} \to \mathbf{y}_{k+N_\beta-\Sigma_{r=j-1}^{j}N_{\beta_r}} \text{ with} \\
\left.\begin{array}{l}
\mathbf{f}^{(\beta_j)}(\mathbf{y}_{k+N_\beta-\Sigma_{r=j}^{n}N_{\beta_r}}, \mathbf{y}_{k+N_\beta-\Sigma_{r=j}^{n}N_{\beta_r}-1}, \mathbf{p}^{(\beta_j)}) = \mathbf{0}, \\
\vdots \\
\mathbf{f}^{(\beta_j)}(\mathbf{y}_{k+N_\beta-\Sigma_{r=j-1}^{j}N_{\beta_r}+2}, \mathbf{y}_{k+N_\beta-\Sigma_{r=j-1}^{j}N_{\beta_r}+1}, \mathbf{p}^{(\beta_j)}) = \mathbf{0}, \\
\mathbf{f}^{(\beta_j)}(\mathbf{y}_{k+N_\beta-\Sigma_{r=j-1}^{j}N_{\beta_r}+1}, \mathbf{y}_{k+N_\beta-\Sigma_{r=j-1}^{j}N_{\beta_r}}, \mathbf{p}^{(\beta_j)}) = \mathbf{0}
\end{array}\right\} \\
\text{for } j = n, n-1, \cdots, 1 \\
P_{\boldsymbol{\varphi}-} : \mathbf{y}_k \to \mathbf{x}'_k \text{ with } \boldsymbol{\varphi}(\mathbf{x}'_k, \mathbf{y}_k, \boldsymbol{\lambda}) = \mathbf{0}; \\
\mathbf{x}'_k = \mathbf{x}_k
\end{array} \tag{6.123}$$

and

$$DP^{(N_\alpha:N_\beta)}(\mathbf{x}_k^*) = DP_{\boldsymbol{\varphi}-}(\mathbf{y}_k^*) \cdot DP^{(N_\beta)}_{\beta-}(\mathbf{y}^*_{k+N_\alpha}) \cdot DP_{\boldsymbol{\varphi}+}(\mathbf{x}^*_{k+N_\alpha}) \cdot DP^{(N_\alpha)}_{\alpha+}(\mathbf{x}_k^*) \tag{6.124}$$

where

$$\begin{array}{l}
DP^{(N_\beta)}_{\beta-}(\mathbf{y}^*_{k+N_\alpha}) = \prod_{i=m}^{1} DP^{(N_{\alpha_i})}_{\alpha_i+}(\mathbf{x}_k^*), \\
DP^{(N_{\alpha_i})}_{\alpha_i+}(\mathbf{x}_k^*) = DP_{\alpha_i+}(\mathbf{x}^*_{k+\Sigma_{r=1}^{i}N_{\alpha_i}-1}) \cdot \ldots \cdot DP_{\alpha_i+}(\mathbf{x}^*_{k+\Sigma_{r=1}^{i-1}N_{\alpha_i}+1}) \cdot DP_{\alpha_i+}(\mathbf{x}^*_{k+\Sigma_{r=1}^{i-1}N_{\alpha_i}}),
\end{array} \tag{6.125}$$

$$\begin{aligned}
DP_{\beta-}^{(N_\beta)}(\mathbf{y}_{k+N_\alpha}^*) &= \prod\nolimits_{j=1}^{m} DP_{\beta_j+}^{(N_{\beta_j})}(\mathbf{y}^*_{k+N_\beta-\Sigma_{r=1}^{j-1}N_{\alpha_i}}),\\
DP_{\beta_j+}^{(N_{\beta_j})}(\mathbf{y}^*_{k+N_\beta-\Sigma_{r=1}^{j-1}N_{\alpha_i}}) &= DP_{\beta-}(\mathbf{x}^*_{k+N_\beta-\Sigma_{r=1}^{j-1}N_{\alpha_i}})\cdot \ldots \cdot DP_{\beta-}(\mathbf{x}^*_{k+N_\beta-\Sigma_{r=1}^{j-1}N_{\alpha_i}-1})\\
&\quad \cdot DP_{\beta-}(\mathbf{x}^*_{k+N_\beta-\Sigma_{r=1}^{j}N_{\alpha_i}}),
\end{aligned} \tag{6.126}$$

$$DP^{(N_\alpha:N_\beta)}(\mathbf{x}_k^*) = \left[\frac{\partial \mathbf{x}'_k}{\partial \mathbf{x}_k}\right]_{\mathbf{x}_k^*}, \tag{6.127}$$

$$\begin{aligned}
DP_{\alpha_i+}^{(N_{\alpha_i})}(\mathbf{x}^*_{k+\Sigma_{r=1}^{i}N_{\alpha_i}}) &= \prod\nolimits_{s=N_{\alpha_i}}^{1}\left[\frac{\partial \mathbf{x}_{k+\Sigma_{r=1}^{i-1}N_{\alpha_i}+s}}{\partial \mathbf{x}_{k+\Sigma_{r=1}^{i-1}N_{\alpha_i}+s-1}}\right]_{\mathbf{x}^*_{k+\Sigma_{r=1}^{i-1}N_{\alpha_i}+s-1}},\\
DP_{\boldsymbol{\varphi}+}(\mathbf{x}^*_{k+N_\alpha}) &= \left[\frac{\partial \mathbf{y}_{k+N_\beta}}{\partial \mathbf{x}_{k+N_\alpha}}\right]_{\mathbf{x}^*_{k+N_\alpha}},\\
DP_{\beta_j-}^{(N_{\beta_j})}(\mathbf{y}^*_{k+N_\beta-\Sigma_{r=1}^{j-1}N_{\beta_j}}) &= \prod\nolimits_{s=1}^{N_{\beta_j}}\left[\frac{\partial \mathbf{y}_{k+N_\beta-\Sigma_{r=1}^{j}N_{\beta_j}-s}}{\partial \mathbf{y}_{k+N_\beta-\Sigma_{r=1}^{j}N_{\beta_j}-s+1}}\right]_{\mathbf{y}^*_{k+N_\beta-\Sigma_{r=1}^{j}N_{\beta_j}-s+1}},\\
DP_{\boldsymbol{\varphi}-}(\mathbf{y}_k^*) &= \left[\frac{\partial \mathbf{x}'_k}{\partial \mathbf{y}_k}\right]_{\mathbf{y}_k^*}.
\end{aligned} \tag{6.128}$$

(i) *The* $(N_\alpha : N_\beta)$ *-synchronization of two sets of discrete dynamical systems* $\cup_{i=1}(P_{\alpha_i}, \mathbf{f}^{(\alpha_i)})$ *and* $\cup_{j=1}(P_{\beta_j}, \mathbf{f}^{(\beta_j)})$ *is persistent if and only if all the eigenvalues* λ_i *(*$i = 1, 2, \cdots, n$*) of* $DP^{(N_\alpha:N_\beta)}(\mathbf{x}_k^*)$ *lie in the unit circles, i.e.,*

$$|\lambda_i|<1 \text{ for } i = 1, 2, \cdots, n. \tag{6.129}$$

(ii) *The* $(N_\alpha : N_\beta)$ *-synchronization of two sets of discrete dynamical systems* $\cup_{i=1}(P_{\alpha_i}, \mathbf{f}^{(\alpha_i)})$ *and* $\cup_{j=1}(P_{\beta_j}, \mathbf{f}^{(\beta_j)})$ *is a saddle-node vanishing if and only if at least one of the real eigenvalues* λ_i *(*$i = 1, 2, \cdots, n_1$ *and* $1 \le n_1 \le n$*) of* $DP^{(N_\alpha:N_\beta)}(\mathbf{x}_k^*)$ *is positive one (+1) and the other eigenvalues are in the unit circle, i.e.,*

$$\lambda_i = 1 \text{ and } |\lambda_j|<1 \text{ for } i, j \in \{1, 2, \cdots, n\} \text{ and } j \ne i. \tag{6.130}$$

(iii) *The* $(N_\alpha : N_\beta)$ *synchronization of two sets of discrete dynamical systems* $\cup_{i=1}(P_{\alpha_i}, \mathbf{f}^{(\alpha_i)})$ *and* $\cup_{j=1}(P_{\beta_j}, \mathbf{f}^{(\beta_j)})$ *is a period-doubling vanishing if and only if at least one of the real eigenvalues* λ_i *(*$i = 1, 2, \cdots, n_1$ *and* $1 \le n_1 \le n$*) of*

$DP^{(N_\alpha:N_\beta)}(\mathbf{x}_k^*)$ is negative one (−1) and the other eigenvalues are in the unit circle, i.e.,

$$\lambda_i = -1 \text{ and } |\lambda_j|<1 \text{ for } i,j \in \{1,2,\cdots,n\} \text{ and } j \neq i. \tag{6.131}$$

(iv) *The $(N_\alpha : N_\beta)$-synchronization of two sets of discrete dynamical systems $\cup_{i=1}(P_{\alpha_i}, \mathbf{f}^{(\alpha_i)})$ and $\cup_{j=1}(P_{\beta_j}, \mathbf{f}^{(\beta_j)})$ is a Naimark vanishing if and only if at least one pair of all the complex eigenvalues $\lambda_i = \alpha_i \pm \beta_i \mathbf{i}$ ($i = 1,2,\cdots,n_1$ and $1 \leq n_1 \leq n/2$) of $DP^{(N_\alpha:N_\beta)}(\mathbf{x}_k^*)$ are on the unit circle and the other eigenvalues are in the unit circle, i.e.,*

$$|\lambda_i| = \sqrt{\alpha_i^2 + \beta_i^2} = 1 \text{ and } |\lambda_j|<1 \text{ for } i,j \in \{1,2,\cdots,n\} \text{ and } j \neq i. \tag{6.132}$$

(v) *The $(N_\alpha : N_\beta)$ synchronization of two sets of discrete dynamical systems $\cup_{i=1}(P_{\alpha_i}, \mathbf{f}^{(\alpha_i)})$ and $\cup_{j=1}(P_{\beta_j}, \mathbf{f}^{(\beta_j)})$ is an ($l_1 : l_2 : l_3$) vanishing if and only if l_1 and l_2 real eigenvalues λ_i of $DP^{(N_\alpha:N_\beta)}(\mathbf{x}_k^*)$ are (−1) and (+1), respectively, and l_3-pairs of complex eigenvalues $\lambda_i = \alpha_i \pm \beta_i \mathbf{i}$ ($i = 1,2,\cdots,n_1$ and $1 \leq n_1 \leq n/2$) of $DP^{(N_\alpha:N_\beta)}(\mathbf{x}_k^*)$ are on the unit circle and the other eigenvalues are in the unit circle, i.e.,*

$$\begin{aligned}
\lambda_i &= -1 \text{ for } i = i_1, i_2, \cdots, i_{l_1} \in \{1,2,\cdots,n\},\\
\lambda_j &= +1 \text{ for } j = j_1, j_2, \cdots, j_{l_2} \in \{1,2,\cdots,n\},\\
|\lambda_r| &= \sqrt{\alpha_r^2 + \beta_r^2} = 1 \text{ for } r = r_1, r_2, \cdots, r_{l_3} \in \{1,2,\cdots,n\},\\
|\lambda_s| &<1 \text{ for } s \in \{1,2,\cdots,n\} \text{ and } s \notin \{i,j,r\}.
\end{aligned} \tag{6.133}$$

(vi) *The $(N_\alpha : N_\beta)$ synchronization of two sets of discrete dynamical systems $\cup_{i=1}(P_{\alpha_i}, \mathbf{f}^{(\alpha_i)})$ and $\cup_{j=1}(P_{\beta_j}, \mathbf{f}^{(\beta_j)})$ is instantaneous if and only if at least one of the eigenvalues λ_i ($i = 1,2,\cdots,n$) of $DP^{(N_\alpha:N_\beta)}(\mathbf{x}_k^*)$ lies out of the unit circle, i.e.,*

$$|\lambda_i|>1 \text{ for } i \in \{1,2,\cdots,n\}. \tag{6.134}$$

Proof The proof can be referred to Luo [2]. □

6.5 An Application of Discrete Systems Synchronization

As in Luo and Guo [4], consider an identical synchronization of the Duffing and Henon maps as an example. The Duffing map is

$$x_{1(k+1)} = x_{2(k)} \text{ and } x_{2(k+1)} = -dx_{1(k)} + cx_{2(k)} - x_{2(k)}^3. \tag{6.135}$$

and the Henon map is

$$y_{1(k+1)} = y_{2(k)} + 1 - ay_{1(k)}^2 \text{ and } y_{2(k+1)} = by_{1(k)}. \tag{6.136}$$

Introduce the vectors as

$$\begin{aligned} &\mathbf{x}_k = (x_{1(k)}, x_{2(k)})^{\mathrm{T}} \text{ and } \mathbf{y}_k = (y_{1(k)}, y_{2(k)})^{\mathrm{T}} \\ &\mathbf{f}^{(\alpha)} = (f_1^{(\alpha)}, f_2^{(\alpha)})^{\mathrm{T}} \text{ for } \alpha = 1, 2. \end{aligned} \tag{6.137}$$

Herein $\alpha = 1$ for the Duffing map and $\alpha = 2$ for the Henon map. Thus, the Duffing map is described by

$$P_1 : \mathbf{x}_k \to \mathbf{x}_{k+1} \text{ and } \mathbf{f}^{(1)}(\mathbf{x}_k, \mathbf{x}_{k+1}, \mathbf{p}^{(1)}) = \mathbf{0}, \tag{6.138}$$

where

$$\begin{aligned} &f_1^{(1)}(\mathbf{x}_k, \mathbf{x}_{k+1}, \mathbf{p}^{(1)}) = x_{1(k+1)} - x_{2(k)}, \\ &f_2^{(1)}(\mathbf{x}_k, \mathbf{x}_{k+1}, \mathbf{p}^{(1)}) = x_{2(k+1)} + dx_{1(k)} - cx_{2(k)} + x_{2(k)}^3; \\ &\mathbf{p}^{(1)} = (c, d)^{\mathrm{T}}. \end{aligned} \tag{6.139}$$

The Henon map is described by

$$P_2 : \mathbf{y}_k \to \mathbf{y}_{k+1} \text{ and } \mathbf{f}^{(2)}(\mathbf{y}_k, \mathbf{y}_{k+1}, \mathbf{p}^{(2)}) = \mathbf{0}, \tag{6.140}$$

where

$$\begin{aligned} &f_1^{(2)}(\mathbf{y}_k, \mathbf{y}_{k+1}, \mathbf{p}^{(2)}) = y_{1(k+1)} - y_{2(k)} - 1 + ay_{1(k)}^2, \\ &f_2^{(2)}(\mathbf{y}_k, \mathbf{y}_{k+1}, \mathbf{p}^{(2)}) = y_{2(k+1)} - by_{1(k)}; \\ &\mathbf{p}^{(2)} = (a, b)^{\mathrm{T}}. \end{aligned} \tag{6.141}$$

Consider the $(N_1 : N_2)$ synchronization of the Duffing and Henon maps with

$$\begin{aligned} &\boldsymbol{\varphi}(\mathbf{x}_k, \mathbf{y}_k, \boldsymbol{\lambda}) = \mathbf{x}_k - \mathbf{y}_k = \mathbf{0}, \\ &\boldsymbol{\varphi}(\mathbf{x}_{k+N_1}, \mathbf{y}_{k+N_2}, \boldsymbol{\lambda}) = \mathbf{x}_{k+N_1} - \mathbf{y}_{k+N_2} = \mathbf{0}, \end{aligned} \tag{6.142}$$

where

$$\begin{aligned} &\mathbf{x}_{k+N_1} = P_1^{(N_1)}\mathbf{x}_k = \underbrace{P_1 \circ P_1 \circ \cdots \circ P_1}_{N_1} \mathbf{x}_k \text{ with} \\ &f_1^{(1)}(\mathbf{x}_{k+i-1}, \mathbf{x}_{k+i}, \mathbf{p}^{(1)}) = x_{1(k+i)} - x_{2(k+i-1)} = 0, \\ &f_2^{(1)}(\mathbf{x}_{k+i-1}, \mathbf{x}_{k+i}, \mathbf{p}^{(1)}) = x_{2(k+i)} + dx_{1(k+i-1)} - cx_{2(k+i-1)} + x_{2(k+i-1)}^3 = 0 \\ &\text{for } i = 1, 2, \cdots, N_1; \end{aligned} \tag{6.143}$$

$$\begin{aligned}
&\mathbf{y}_{k+N_2} = P_2^{(N_2)}\mathbf{y}_k = \underbrace{P_2 \circ P_2 \circ \cdots \circ P_2}_{N_2} \mathbf{y}_k \text{ with} \\
&f_1^{(2)}(\mathbf{y}_{k+j}, \mathbf{y}_{k+j-1}, \mathbf{p}^{(2)}) = y_{1(k+j)} - y_{2(k+j-1)} - 1 + a y_{1(k+j-1)}^2 = 0, \\
&f_2^{(2)}(\mathbf{y}_k, \mathbf{y}_{k+1}, \mathbf{p}^{(2)}) = y_{2(k+j)} - b y_{1(k+j-1)} = 0 \\
&\text{for } j = 1, 2, \cdots, N_2.
\end{aligned} \tag{6.144}$$

For the $(N_1 : N_2)$ synchronization, the equivalent mapping structure is

$$\mathbf{x}'_k = P_{\boldsymbol{\varphi}-} \circ P_{2-}^{(N_2)} \circ P_{\boldsymbol{\varphi}+} \circ P_{1+}^{(N_1)} \mathbf{x}_k. \tag{6.145}$$

If $\mathbf{x}'_k = \mathbf{x}_k$, we have

$$\begin{aligned}
&\left.\begin{aligned}
&f_1^{(1)}(\mathbf{x}_{k+i-1}, \mathbf{x}_{k+i}, \mathbf{p}^{(1)}) = x_{1(k+i)} - x_{2(k+i-1)} = 0 \\
&f_2^{(1)}(\mathbf{x}_{k+i-1}, \mathbf{x}_{k+i}, \mathbf{p}^{(1)}) = x_{2(k+i)} + d x_{1(k+i-1)} - c x_{2(k+i-1)} + x_{2(k+i-1)}^3 = 0
\end{aligned}\right\} \\
&\text{for } i = 1, 2, \cdots, N_1; \\
&\boldsymbol{\varphi}(\mathbf{x}_{k+N_1}, \mathbf{y}_{k+N_2}, \boldsymbol{\lambda}) = \mathbf{x}_{k+N_1} - \mathbf{y}_{k+N_2} = \mathbf{0}; \\
&\left.\begin{aligned}
&f_1^{(2)}(\mathbf{y}_{k+j}, \mathbf{y}_{k+j-1}, \mathbf{p}^{(2)}) = y_{1(k+j)} - y_{2(k+j-1)} - 1 + a y_{1(k+j-1)}^2 = 0 \\
&f_2^{(2)}(\mathbf{y}_k, \mathbf{y}_{k+1}, \mathbf{p}^{(2)}) = y_{2(k+j)} - b y_{1(k+j-1)} = 0
\end{aligned}\right\} \\
&\text{for } j = N_2, \cdots, 2, 1; \\
&\boldsymbol{\varphi}(\mathbf{x}_k, \mathbf{y}_k, \boldsymbol{\lambda}) = \mathbf{x}_k - \mathbf{y}_k = \mathbf{0}.
\end{aligned} \tag{6.146}$$

From which the fixed points of Eq. (6.145) [i.e., $\mathbf{x}_{k+i}^*$ $(i = 1, 2, \cdots, N_1)$ and $\mathbf{y}_{k+j}^*$ $(j = 1, 2, \cdots N_2)$] can be obtained. The corresponding stability boundary of such fixed points is given the eigenvalue analysis, i.e.,

$$\Delta\mathbf{x}'_k = DP_{\boldsymbol{\varphi}+} \cdot DP_{2-}^{(N_2)} \cdot DP_{\boldsymbol{\varphi}+} \cdot DP_{1+}^{(N_1)} \Delta\mathbf{x}_k; \tag{6.147}$$

where

$$\begin{aligned}
DP_{\boldsymbol{\varphi}-}(\mathbf{y}_k^*) &= \left[\frac{\partial \mathbf{x}'_k}{\partial \mathbf{y}_k}\right]_{\mathbf{y}_k^*} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \\
DP_{2-}^{(N_2)} &= \prod\nolimits_{j=1}^{N_2} DP_{2-}(\mathbf{y}_{k+j}^*), \\
DP_{2-}(\mathbf{y}_{k+j}^*) &= \left[\frac{\partial \mathbf{y}_{k+j-1}}{\partial \mathbf{y}_{k+j}}\right]_{\mathbf{y}_{k+j}^*} = -\frac{1}{b}\begin{bmatrix} 0 & 1 \\ b & 2a y_{1(k+j-1)}^* \end{bmatrix}; \\
DP_{\boldsymbol{\varphi}+}(\mathbf{x}_{k+N_1}^*) &= \left[\frac{\partial \mathbf{y}_{k+N_2}}{\partial \mathbf{x}_{k+N_1}}\right]_{\mathbf{x}_{k+N_1}^*} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \\
DP_{1+}^{(N_1)} &= \prod\nolimits_{i=N_1}^{1} DP_{2-}(\mathbf{x}_{k+j}^*), \\
DP_{1+}(\mathbf{x}_{k+j-1}^*) &= \left[\frac{\partial \mathbf{x}_{k+j}}{\partial \mathbf{x}_{k+j-1}}\right]_{\mathbf{x}_{k+j-1}^*} = \begin{bmatrix} 0 & 1 \\ -d & -c + 3(x_{2(k+j-1)}^*)^2 \end{bmatrix}.
\end{aligned} \tag{6.148}$$

Through the above analysis procedure, the $(N_1 : N_2)$ synchronization domains and boundaries can be determined from Theorem 6.7. In Eq. (6.145), we can form a new map iteration

$$\begin{aligned} &\mathbf{x}_{J+1} = P\mathbf{x}_J \text{ with} \\ &\mathbf{x}_J \equiv \mathbf{x}_k \text{ and } P \equiv P_{\boldsymbol{\varphi}-} \circ P_{2-}^{(N_2)} \circ P_{\boldsymbol{\varphi}+} \circ P_{1+}^{(N_1)}. \end{aligned} \tag{6.149}$$

Using Eq. (6.149), numerical iteration can be done to observe the $(N_1 : N_2)$ identical synchronization of the Duffing and Henon maps.

As in Luo and Guo [4], consider parameters of $a = 0.8$, $c = 2.75$ and $d = 0.2$. From the mapping in Eq. (6.149), the $(1:1)$ -identical synchronization of the Duffing and Henon maps is simulated, as shown in Fig. 6.9. The bifurcation scenario alike plots for $x_{1(k)}$ and $x_{2(k)}$ with $y_{1(k)}$ and $y_{2(k)}$. The shaded regions are for the $(1:1)$ synchronization. PD and SN represent period-doubling and saddle-node vanishing of the $(1:1)$ synchronization, respectively. The synchronization range is $b \in (-\infty, -30.84)$ and $b \in (33.88, \infty)$ in Fig. 6.9a, b. In Fig. 6.9c–f, the zoomed view for small parameter ranges are presented. The parameter ranges are given by $b \in (1.2431,\ 1.3687)$ and $b \in (-1.7667, -1.4216)$, respectively. The analytical predictions of the $(1:1)$-synchronization is presented in Fig. 6.9. The solid curves are the $(1:1)$ synchronizations. PD and SN represent period-doubling and saddle-node vanishing of the $(1:1)$ synchronization, respectively. The instantaneous $(1:1)$ synchronizations are represented by dashed curves. For numerical simulations, the instantaneous synchronization state cannot be achieved. The $(1:1)$ synchronization given by the analytical prediction matches with the numerical prediction. The large parameter ranges for the $(1:1)$ synchronization are presented in Fig. 6.10a, b. The small parameter ranges for the $(1:1)$-synchronization are arranged in Fig. 6.10c–f. The corresponding parameter maps for $(1:1)$-synchronization are presented in Fig. 4.13. The shaded regions are for the $(1:1)$ synchronization. PD and SN represent period-doubling and saddle-node vanishing of the $(1:1)$ synchronization, respectively. The intersected points of the PD and SN vanishing are $(1, 1, 0)$-critical synchronization vanishing with $\lambda_1 = -1$ and $\lambda_2 = 1$. Figure 6.11a, c, e is for overall parameter maps, and Fig. 6.11b, d, f is for the zoomed views of parameter maps. Figure 6.11a, b shows parameter map (a, b) for $c = 2.75$ and $d = 0.2$. Figure 6.11c, d presents the parameter maps (d, b) for $a = 0.8$ and $c = 2.75$. Figure 6.11e, f gives the parameter (c, b) for $a = 0.8$ and $d = 0.2$. For the parameter maps, the $(1:1)$ synchronizations exist in different regions with many cusp points, and such cusp points will be very difficult to be analyzed by the catastrophe analysis. Other discrete dynamical system synchronization can be carried out from the theory of discrete dynamical system synchronization, which is presented in this chapter.

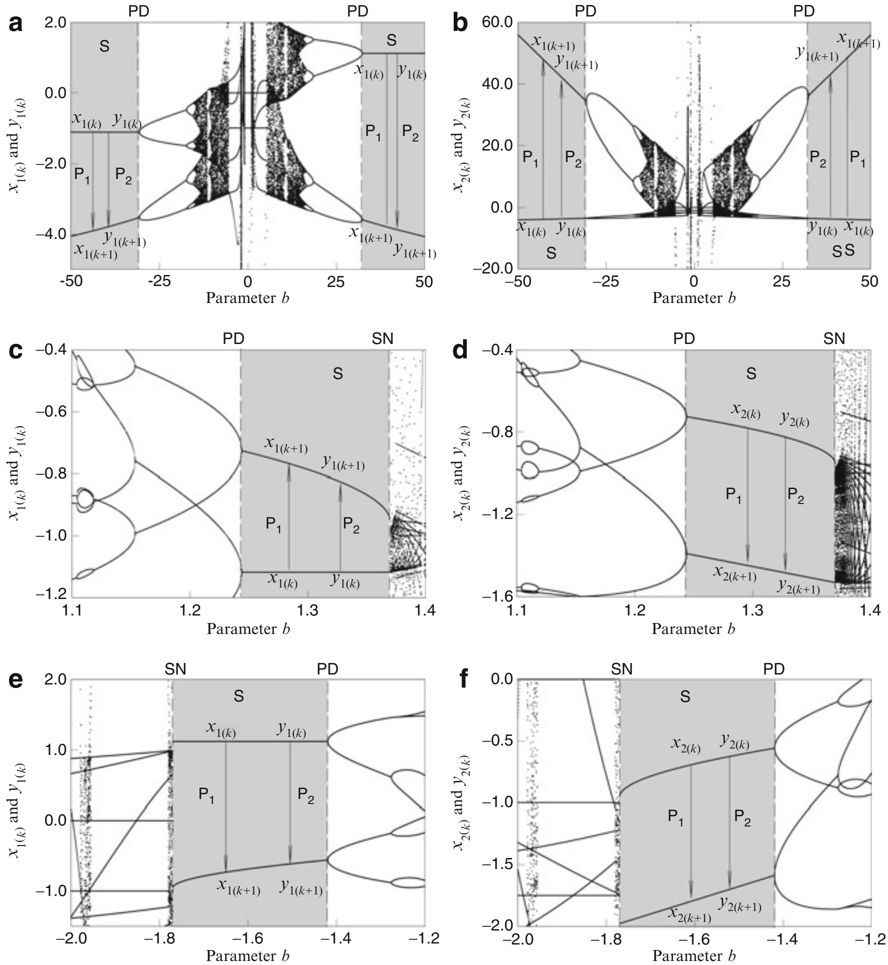

Fig. 6.9 The numerical iteration for the (1:1) synchronization of two discrete dynamical systems with the Duffing and Henon maps. Bifurcation scenario alike plots for $x_{1(k)}$ and $x_{2(k)}$ with $y_{1(k)}$ and $y_{2(k)}$: (**a**) and (**b**) for $b \in (-\infty, -30.84)$ and $b \in (33.88, \infty)$; (**c**) and (**d**) for $b \in (1.2431,\ 1.3687)$; (**e**) and (**f**) for $b \in (-1.7667, -1.4216)$. The *shaded regions* are for the (1:1) synchronization. PD and SN represent period-doubling and saddle-node vanishing of the (1:1) synchronization, respectively ($a = 0.8$, $c = 2.75$ and $d = 0.2$)

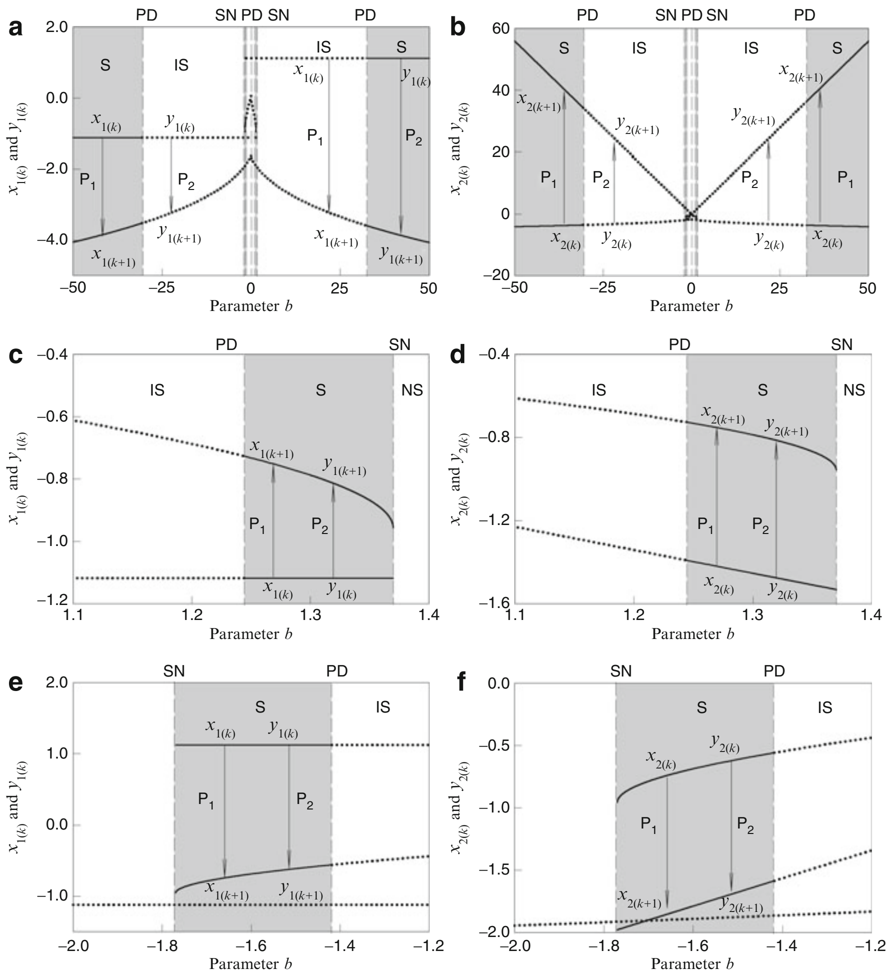

Fig. 6.10 The analytical prediction of the (1:1) synchronization of two discrete dynamical systems with the Duffing and Henon maps. The iterative states $x_{1(k)}$ and $x_{2(k)}$ with $y_{1(k)}$ and $y_{2(k)}$ are presented: (**a**) and (**b**) for $b \in (-\infty, -30.84)$ and $b \in (33.88, \infty)$; (**c**) and (**d**) for $b \in (1.2431, 1.3687)$; (**e**) and (**f**) for $b \in (-1.7667, -1.4216)$. The *shaded regions* are for the (1:1) synchronization. PD and SN represent period-doubling and saddle-node vanishing of the (1:1) synchronization, respectively. The instantaneous (1:1) synchronizations are represented by the *dotted curves* ($a = 0.8$, $c = 2.75$, and $d = 0.2$)

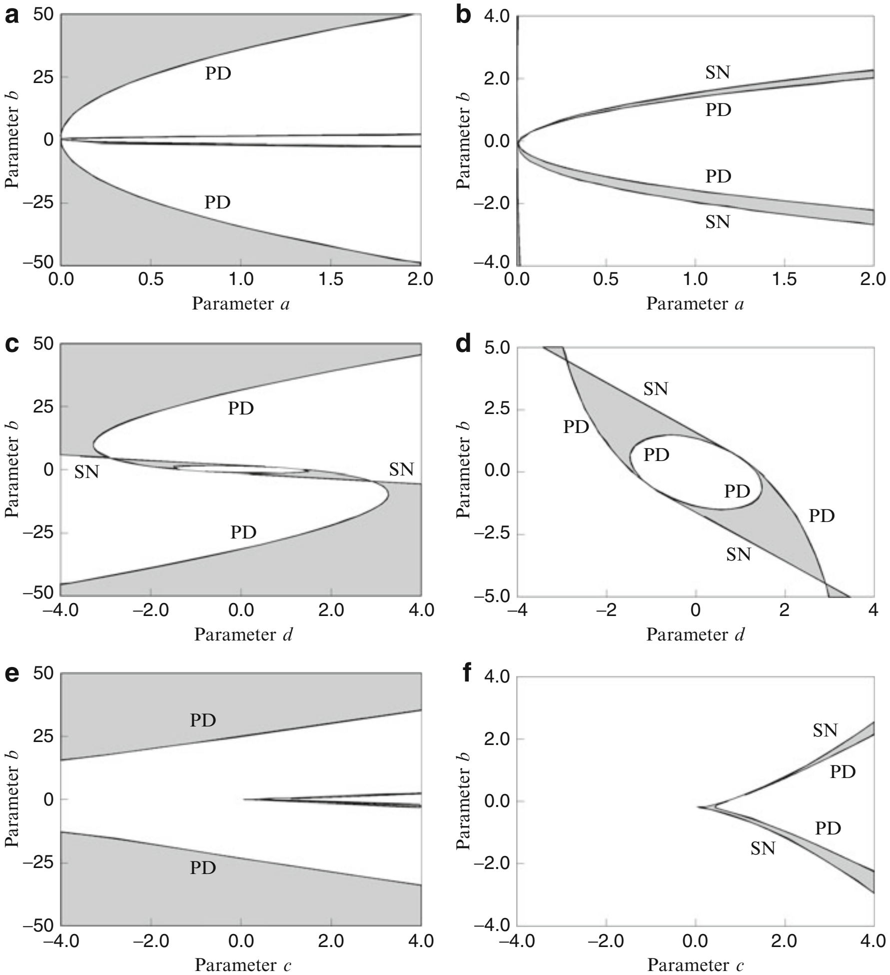

Fig. 6.11 Parameter maps of the $(1:1)$ synchronization of two discrete dynamical systems with the Duffing and Henon maps: (**a**) and (**b**) parameter map (a, b) for $c = 2.75$ and $d = 0.2$; (**c**) and (**d**) parameter maps (d, b) for $a = 0.8$ and $c = 2.75$; (**e**) and (**f**) parameter (c, b) for $a = 0.8$ and $d = 0.2$. The overall views are given on the *left-hand side*, and the zoomed views are given on the *right-hand side*. The *shaded regions* are for the $(1:1)$ synchronization. PD and SN represent period-doubling and saddle-node vanishing of the $(1:1)$ synchronization, respectively

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